The Limit Order Book and the Break-Even Conditions Revisited

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The Rock-Glosten break-even conditions state that offer prices and bid prices equal ``upper tail'' and ``lower tail'' expectation, respectively, where the conditioning is on the order flow. This paper shows that the intuition that underlies these conditions is incomplete: In some relevant economic environments (e.g., the ``uniform example'' considered in Glosten (1994)), the book that satisfies the break-even conditions is the limit of an artificial market game. In this artificial game, strategic liquidity suppliers must offer a predetermined number of shares at a predetermined price interval in a manner that minimizes expected gains.

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Abstract. The Rock-Glosten break-even conditions state that offer prices and bid prices equal "upper tail" and "lower tail" expectation, respectively, where the conditioning is on the order flow. This paper shows that the intuition that underlies these conditions is incomplete: In some relevant economic environments (e.g., the "uniform example" considered in Glosten (1994)), the book that satisfies the break-even conditions is the limit of an artificial market game. In this artificial game, strategic liquidity suppliers must offer a predetermined number of shares at a predetermined price interval in a manner that minimizes expected gains.

The assumption that liquidity suppliers break even provides a unifying framework for the literature on market microstructure. The break even conditions state that prices equal the expected value of the asset conditional on execution.

In a market organized as a pure limit order book, a market order walks the book, picking off limit orders at their limit-prices. In that context, the liquidity suppliers are the limit order submitters. Following Rock (1990) and Glosten (1994), it is common to express the break even conditions in terms of upper and lower tail expectations of the asset value, where the conditioning is on the order flow.

The goal of this paper is to reassess the Rock-Glosten break-even conditions. To that end, we study a model of imperfect competition in liquidity provision and ask whether the competitive equilibrium is the limit of the model with a large number of liquidity suppliers.

We consider economic environments where the profitability of a marginal offer depends only on its limit price and the cumulative depth up to that price. We model the interaction between the liquidity suppliers as a static game, as in Bernhardt and Hughson (1997) and Biais, Martimort, and Rochet (2000). We then use the principle of optimality to examine the problem of a liquidity supplier.[1] The Bellman equation can be used to construct a candidate for equilibrium, and we see that the candidate book satisfies, at the limit when the number of liquidity suppliers is large, the

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[1]The problem is not dynamic in time. We just exploit the structure of the problem and use the cumulative depth as the state variable.

Rock-Glosten conditions. However, it is not clear that the candidate is indeed an equilibrium.

The problem of a liquidity supplier is locally linear, which implies that the Bellman equation is degenerate. That is, we don't know whether the candidate for equilibrium is an outcome of a maximization or a minimization effort. We use Green's theorem to state sufficient conditions for the candidate to be an equilibrium. However, these sufficient conditions are not merely technical, and in fact in some relevant economic environments the conditions are not satisfied.

This leads us to consider an "artificial" market game, one where the goal of the liquidity suppliers is to minimize gains. In the artificial game, each liquidity suppliers must offer a prespecified number of shares in a prespecified price interval. The aggregate number of shares that must be offered equals the number of shares that are offered in the competitive equilibrium.

In both market games, the realistic and the artificial, liquidity suppliers weigh in a similar manner the benefits and risks of placing aggressive offers. In the realistic market game, a liquidity supplier is reluctant to offer at aggressive prices because of the adverse selection risk. At the same time, the high precedence (thanks to the price priority rule) increases the likelihood of trading against uninformed orders. The optimal strategy balances this trade-off.

In the artificial market game, the tradeoff is mirrored: a liquidity supplier is also reluctant to offer at aggressive prices, only that this time it is because of the potential gains from trading against uninformed orders. An aggressive offer, on the other hand, has the potential for large losses to informed orders. The optimal strategy in the artificial game balances this trade-off.

We consider several examples, in particular the two examples Glosten (1994) considers: the exponential-normal example and the uniform example. In the exponential-normal example, we show that the Bellman equation characterizes the equilibrium of the realistic game. In contrast, in the uniform example, the Bellman equation characterizes the equilibrium of the artificial game. In particular, the competitive book that Glosten (1994) reports is the limit of the equilibrium book in the artificial game.

It is possible that the Rock-Glosten conditions are satisfied at the limit of equilibria of the artificial trading game, computed via the Bellman equation, but that the conditions are also satisfied at the limit of equilibria of the realistic game, even though we may not know how to compute those equilibria. In that case, we should question the break-even conditions paradigm.

An alternative explanation is that in some economic environments the break even conditions should not be expressed in terms of expectation conditional on the order flow, as Rock (1990) and Glosten (1994) posit. Baruch (2008) studies a variant of the uniform example and finds an equilibrium with random limit orders. When limit orders are random, there is an additional uncertainty about the execution of a limit order. In other words, conditional on execution, a liquidity supplier cannot infer whether the incoming market order is large or the book is thin. Indeed, the limit of equilibria in Baruch (2008) does not satisfy the Rock-Glosten conditions.

The paper is organized as follows. In Section 1 we describe the economic environment. In Section 2 we state the Rock-Glosten break even conditions. In Section 3 we define the equilibrium, and in Section 4 we use the Bellman equation to derive a candidate for equilibrium. In Section 5 we provide sufficient conditions for equilibrium and present two examples. In Section 6 we define the artificial game and present examples. In Section 7 we conclude.

1. The Economic Environment

We model a pure limit order market for a single risky asset. There are two types of traders in the model: risk neutral, uninformed liquidity suppliers and a single active trader. Initially, the liquidity suppliers populate the limit order book. Those limit orders are then prioritized by their limit prices (i.e., price priority rule is enforced). Next, the active trader observes all the bids and offers in the book and submits a market order. Based on their priority, limit orders are matched with the market order until the market order is completly filled (i.e., the market order "walks the book"), and trade takes place at the limit prices (i.e., trading against limit orders is discriminatory). Following the trade, the asset is liquidated; the liquidation value is $\widetilde{v}$. Without loss of generality, we focus on the offer side of the book.[2]

The ask price is the smallest price at which offers are posted. Let $y(\cdot)$ denote the supply function; i.e., the liquidity suppliers offer $y(x)$ shares up to the price x. By definition, $y(\cdot)$ is positive and increasing, and it is strictly positive above the ask price and strictly increasing at all those prices that are populated with offers. To emphasize that the active trader observes the supply function, we denote the order size by $\widetilde{\delta}(y(\cdot))$.

[2]We don't lose generality because the liquidity suppliers are uninformed.

The tilde notation indicates that the order also depends on the trader's unobservable characteristics (e.g. position in the asset) and possibly private information about the liquidation value.

Liquidity suppliers take into account the active trader's behavior when posting their offers. In particular, given an arbitrary supply function, $y(\cdot)$, they understand the joint distribution of the asset random value, $\widetilde{v}$, and the active trader's order size, $\widetilde{\delta}(y(\cdot))$.

Consider a marginal offer at a price x when the supply function is $y(\cdot)$. The price priority rule implies that the offer is executed only if $\widetilde{\delta}(y(\cdot)) > y(x)$. Because limit orders are filled at their limit prices, it follows that the ex post profitability of a marginal offer at a price x is given by:[3]

$$(x - \tilde{v}) I_{\{\widetilde{\delta}(y(\cdot)) > y(x)\}}$$

and the ex ante profitability is

$$U(x, y(\cdot)) = E(x - \tilde{v}) I_{\{\widetilde{\delta}(y(\cdot)) > y(x)\}} = \left(x - E\left[\widetilde{v}\,\middle|\,\widetilde{\delta}(y(\cdot)) > y(x)\right]\right) \text{Prob}(\widetilde{\delta}(y(\cdot)) > y(x))$$

In this paper we restrict attention to economic environments in which the profitability of a marginal limit order depends only on its offer price and the cumulative number of shares offered up to that price. More specifically, there are two continuous functions, defined on the the positive quadrant, $\epsilon(\cdot, \cdot)$ and $u(\cdot, \cdot)$, such that for every pair $(x, y(\cdot))$, we have $EI_{\{\widetilde{\delta}(y(\cdot)) > y(x)\}} = \epsilon(x, y(x))$ and $U(x, y(\cdot)) = u(x, y(x))$. We refer to $u(x, y)$ as the profitability of a marginal offer at (x, y); i.e. the profitability of a marginal offer at x when y shares are offered at prices better than x. Similarly, $\epsilon(x, y)$ is the probability of execution of a marginal offer at (x, y). Let $\mathcal{D} = \{(x, y) : \epsilon(x, y) > 0\}$ be the "relevant region:" marginal offers at $(x, y) \notin \mathcal{D}$ are never filled up. We assume that $u(\cdot, \cdot)$ is smooth in the relevant region. Intuitively, the smoothness requirement means that the distribution functions of all random variables that underly the economic environment (e.g. asset value) are smooth.

We conclude this section with some examples. Each example illustrates an economic environment where we can calculate explicitly the function $u(\cdot, \cdot)$ and the "relevant" region D.

The Exponential Example:

[3]Because $y(x)$ is the number of shares offered up to x, it follows that if $y(\cdot)$ has jump at x, then the expression reflects the profitability of the "first unit" offered at x. This paper studies equilibria with continuous supply functions.

The liquidation value of the asset is $\widetilde{v} = \widetilde{s} + \widetilde{\epsilon}$. An active trader with an exponential utility observes the signal $\widetilde{s}$. The trader's initial position in the asset is $\widetilde{I}$. The noise term, $\widetilde{\epsilon}$, is a zero mean normal random variable with variance σ^2, and $\widetilde{\epsilon}$ is independent of $\widetilde{s}$ and $\widetilde{I}$.

Given all the bids and offers in the book, let $t(\delta)$ denote the marginal price of an order of size δ. That is, if $\delta < 0$, then $t(\delta)$ is the proceeds for selling the last fraction of the order, and if $\delta > 0$ then $t(\delta)$ is the cost of the last fraction. Because of the price priority rule, the function $t(\cdot)$ is non-decreasing, and therefore $\int_0^\delta t(z)dz$ is convex.

The trader infers $t(\cdot)$ from $y(\cdot)$ and submits a market order of size δ that maximizes[4]

$$E\left[-\exp(-\gamma\widetilde{W})\Big| I, s\right]$$

where $\widetilde{W} = (\delta + I)(s + \widetilde{\epsilon}) - \int_0^\delta t(z)dz$. Because of the exponential utility function, the investor's problem is equivalent to the concave problem:

$$\max_\delta \theta\delta - \gamma\frac{\sigma^2}{2}\delta^2 - \int_0^\delta t(z)dz$$

where $\theta = s - \gamma\sigma^2 I$.[5] If the optimal size turns out to be strictly positive, then it equals $\sup\{\delta : \theta - \gamma\sigma^2\delta - t(\delta) > 0\}$, or equivalently, the order walks up the book as long as $\theta > \gamma\sigma^2 y(x) + x$.[6]

The liquidity suppliers don't observe θ, but they understand the active trader's strategy. Let $F(\cdot)$ be the distribution function of $\widetilde{\theta}$, $v(\theta) = E[\widetilde{v}|\widetilde{\theta} = \theta]$, and $v^+(\theta) = E[\widetilde{v}|\widetilde{\theta} > \theta]$. The profitability of a marginal offer is then

$$\begin{aligned} u(x,y) &= E(x - \widetilde{v})I_{\{\widetilde{\theta} > \gamma\sigma^2 y + x\}} \\ &= x\left(1 - F\left(\gamma\sigma^2 y + x\right)\right) - \int_{x+y}^\infty v(\theta)dF(\theta) \\ &= \left(x - v^+\left(\gamma\sigma^2 y + x\right)\right)\left(1 - F\left(\gamma\sigma^2 y + x\right)\right). \end{aligned}$$

In Biais, Martimort, and Rochet (2000), the support of $\widetilde{s}$ and $\widetilde{I}$ are are closed intervals. The relevant region is $\mathcal{D} = \{(x,y) : x + \gamma\sigma^2 y < \bar{\theta}\}$, where $\bar{\theta}$ is the upper support of $\widetilde{\theta}$. The probability of execution is $\epsilon(x,y) = 1 - F\left(\gamma\sigma^2 y + x\right)$.

[4]Wherever $y(\cdot)$ is invertible (i.e., prices at which the supply function is strictly monotone), its inverse is $t(\cdot)$.

[5]In Glosten's (1994) terminology, the informed trader's marginal valuation is $\theta - \gamma\sigma^2\delta$.

[6]The objective function of the concave problem may not be smooth because $t(\cdot)$ may not be continuous (for example, if the spread is strictly positive then at zero $t(\cdot)$ has a point of discontinuity). The argument we invoke is that the optimal size is greater than any size at which the derivative of the objective function is strictly positive. And this is true because the problem is concave.

In Glosten (1994), $\widetilde{s}$ and $\widetilde{I}$ are assumed to be independent and normally distributed, and in particular $\mathcal{D}$ is the positive quadrant.

The Uniform Example:

We now consider an economic environment in which with probability μ the active trader is an uninformed trader who wants to buy at most $\widetilde{k}$ shares up to the reservation price $\widetilde{p}$. With probability $1-\mu$ the active trader is an informed trader. In that case, the trader observes the true value of the asset, $\widetilde{v}$, and picks off all offers below $\widetilde{v}$ and all bids above $\widetilde{v}$.

The ex ante profitability of a marginal offer is therefore

$$\begin{aligned} u(x,y) &= E\left[(1-\mu)(x-\widetilde{v})I_{\{\widetilde{v}>x\}} + \mu(x-E\widetilde{v})I_{\{\widetilde{p}>x,\widetilde{k}>y\}}\right] \\ &= (1-\mu)E(x-\widetilde{v})I_{\{\widetilde{v}>x\}} + \mu(x-E\widetilde{v})EI_{\{\widetilde{p}>x,\widetilde{k}>y\}} \end{aligned}$$

In the uniform example in Glosten (1994), the liquidation value is uniformly distributed over $[-L,L]$, and the joint distribution of $\widetilde{p}$ and $\widetilde{k}$ satisfies [7]

$$EI_{\{\widetilde{p}>x,\widetilde{k}>y\}} = \begin{cases} \frac{L-(x+y)}{2L} & x+y \le L \\ 0 & x+y > L \end{cases}$$

Hence, in the uniform example, the ex ante expected profit of a marginal offer is given by

$$u(x,y) = \begin{cases} -\frac{1-\mu}{4L}\left(L-x\right)^2 + \frac{\mu}{2L}x(L-(x+y)) & x+y \le L \\ 0 & x+y > L \end{cases} \tag{1}$$

The relevant region is $\mathcal{D} = \{(x,y) : x+y < L\}$, and the probability of execution is

$$\epsilon(x,y) = (1-\mu)\frac{L-x}{2L} + \mu\frac{L-(x+y)}{2L}$$

The Inelastic Demand Example:

In this economic environment, the size of the active trader's order is a sufficient statistic about the asset value. Let $F(\delta)$ be the cumulative distribution function of

[7] In other words, an uninformed trader's order walks up the book as long as $\widetilde{\epsilon} > x + y(x)$ where $\widetilde{\epsilon}$ is uniformly distributed $[-L,L]$. In Glosten's terminology, the marginal valuation of an uninformed trader is $\epsilon - y(x)$.

the order size, $v(\delta) = E[\tilde{v}|\widetilde{\delta} = \delta]$, and $v^+(\delta) = E[\tilde{v}|\widetilde{\delta} > \delta]$. The expected profit of a marginal offer is

$$\begin{aligned} u(x,y) &= E(x - \widetilde{v})I_{\{\widetilde{\delta}>y\}} = x(1 - F(y)) - \int_y^\infty v(u)dF(u) \\ &= (x - v^+(y))(1 - F(y)) \end{aligned}$$

The relevant region is $\mathcal{D} = \{(x,y) : F(y) < 1\}$, and the probability of execution is $\epsilon(x,y) = (1 - F(y))$.

2. The Break Even Conditions

Following Rock (1990) and Glosten (1994), it is common to posit that when there are infinitely many liquidity suppliers and it is costless to post offers then the supply function satisfies the break even conditions:

$$x \leq E\left[\widetilde{v}|\widetilde{\delta}(y(\cdot)) > y(x)\right]$$

with equality whenever $dy(x) > 0$. The right hand side is commonly called the "upper tail expectation." We may have a strict inequality because there are prices at which even competitive liquidity suppliers do not offer shares (e.g., prices that are below the ask price).

An economic environment is said to be *regular* if: (i) There is a unique supply function, which we denote by $y_\infty(\cdot)$, that satisfies the break even condition, and (ii) In the relevant region $\mathcal{D}$, the sign of $u(\cdot,\cdot)$ is negative above the graph of $y_\infty(x)$ and positive below the graph.[8] We denote by ask_∞ the ask price that corresponds to the supply function $y_\infty(\cdot)$.

Though our analysis is general, we are only interested in regular economic environments where the argument for the break even conditions sounds appealing. In a regular economic environment, the following procedure can be used to construct the supply function $y_\infty(\cdot)$: ask_∞ is the smallest positive root of $x \to u(x,0)$. For all $x < ask_\infty$, we set $y_\infty(x) = 0$. For all $x > ask_\infty$, we set $y_\infty(x)$ to be the value where the function $y \to u(x,y)$ changes signs.

[8]The graph of $y_\infty(\cdot)$ is a subset of the positive quadrant: it is the collection of all the pairs $(x, y_\infty(x))$.

3. Equilibrium with n Strategic Liquidity suppliers

We follow the modeling choice of Bernhardt and Hughson (1997) and Biais, Martimort, and Rochet (2000), and model the interaction between the liquidity suppliers as a game with n strategic liquidity suppliers who post offers simultaneously.

For technical reasons, we make the assumption that there is an exogenous maximum price X at which offers can be posted. With X we associate

$$Y = \operatorname*{argmax}_y \int^y u(X,u)du \tag{2}$$

In a regular economic environment, if $X < ask_\infty$, then no offers will be posted. If $X > ask_\infty$ and (X,Y) is in the relevant region $\mathcal{D}$, then the profit of a marginal offer at (X,Y) is zero; i.e., without incurring a loss, liquidity providers cannot offer more than Y shares at X.[9]

We let $q_i(\cdot)$ denote the marginal supply function of the i-th liquidity supplier, and set $q_{-i}(\cdot) = \sum_{j\neq i} q_j(\cdot)$: In the price interval $(x, x+dx)$, the i-th liquidity supplier offers $q_i(x)dx$ shares. If $y(x)$ shares are offered by all liquidity suppliers at prices up to x, then the ex ante profit is $u(x,y(x))q_i(x)dx$ and at $x+dx$, the total number of shares offered is $y(x+dx) = y(x) + (q_i(x) + q_{-i}(x))dx$. This means that the problem of the i-th liquidity supplier can be written as

$$\begin{cases} \max\limits_{q(x)\geq 0} \int_0^X u(x,y(x))q(x)dx \\ \text{subject to} \\ dy(x) = (q_{-i}(x) + q(x))dx \\ y(0) = 0 \end{cases} \tag{3}$$

Implicit in the the setting of the problem is the assumption that no discrete offers are posted. However, we don't impose any upper bound on $q(x)$. Thus, if it is optimal to offer a discrete offer, then we will not be able to find a solution to this maximization problem.

An equilibrium is an n-tuple of functions $(q_1(\cdot), q_2(\cdot), \ldots, q_n(\cdot))$ such that for every i, $q_i(\cdot)$ solves problem (3). A symmetric equilibrium is an equilibrium in which all liquidity suppliers use the same marginal supply function.

[9]We guess that in a model with endogenous maximum price, X would be the highest price at which the supply function $y_\infty(\cdot)$ is strictly increasing.

4. The Bellman Equation

In this section, we use the Bellman equation to informally derive a necessary condition for an equilibrium in which for all i, $q_i(\cdot)$ is finite and strictly positive above the ask price. Even though the problem of a liquidity supplier is static, we can nevertheless use the principle of optimality, as if a liquidity supplier first chooses how many shares to offer at aggressive prices.

Define

$$v(x_0, y_0) = \begin{cases} \max\limits_{q(x)\geq 0} \int_{x_0}^{X} u(x,y)q(x)dx \\ \text{subject to} \\ dy = q_{-i}(x) + q(x)dx \\ y(x_0) = y_0 \end{cases}$$

The Bellman equation is

$$\max_{q\geq 0} v_x + v_y(q_{-i}(x) + q) + u(x,y)q = 0.$$

Because the problem is linear in q, we must have

$$\begin{aligned} v_y =& - u(x,y) \\ v_x =& - v_y q_{-i}(x) = q_{-i}(x)u(x,y) \end{aligned}$$

Now, differentiate the first equation by x and the second by y, to get

$$\begin{aligned} v_{yx} =& - u_x(x,y) \\ v_{xy} =& q_{-i}(x)u_y(x,y) \end{aligned}$$

So, for the trader to have an optimal positive solution, the equilibrium supply function must solve the equation

$$q_{-i}(x) = -\frac{u_x(x,y(x))}{u_y(x,y(x))}$$

This must hold for every trader, and moreover the right hand side is independent of i.

We conclude that the equilibrium is symmetric, and the supply function solves the o.d.e.

$$\frac{dy}{dx} = -\frac{n}{n-1}\frac{u_x(x,y)}{u_y(x,y)}, \quad y(X) = Y \tag{4}$$

in the interval $[ask, X]$. In a regular economic environment, the boundary condition is natural: to offer less than Y shares would mean to leave money on the table, while to offer more than Y would imply losses.

Note that if we denote by $y_n(\cdot)$ the solution to (4), then intuitively we see that $y_n(\cdot)$ is an increasing sequence of functions (i.e., $y_n(x) \leq y_{n+1}(x)$), and $\lim_{n\to\infty} y_n(\cdot)$ is the solution to the o.d.e.[10]

$$\frac{dy}{dx} = -\frac{u_x(x,y)}{u_y(x,y)}, \quad y(X) = Y$$

which is given implicitly by equation $u(x, y(x)) = 0$; i.e., the break even conditions are satisfied at the limit.

Before leaving this section, we make the following observation. Nowhere, in the analysis above, have we used the fact that the objective is to maximize gains, rather than say minimize them. Moreover, because the Bellman equation is linear in q, there are no second order conditions that can be checked. The standard approach in dynamic programming is to use the value function to verify the solution. Here, however, the value function is only identified along the "equilibrium path." That is, we know $v(x, y(x))$, rather than $v(x, y)$ for arbitrary pairs of (x, y). We therefore have to use a different approch to verify the equilibrium. We do so in the next section.

5. A verification Theorem

Given a strictly monotone solution to the o.d.e. (4), we want to verify that the solution indeed defines the equilibrium. Let ask_n be the first price, when we solve the o.d.e. (4) backward, at which the solution intercepts the x axis. Let $y^*(\cdot)$ denote the solution to the backward differential equation (4) in the interval $[ask_n, X]$. For $x < ask_n$, set $y^*(x) = 0$. Let

$$q^*(x) = \begin{cases} 0, & x < ask_n \\ -\frac{1}{n-1}\frac{u_x(x,y^*(x))}{u_y(x,y^*(x))}, & ask_n \leq x \leq X \end{cases}$$

so that $y^*(x) = \int_0^x nq^*(u)du$. Finally, define Miele's fundamental function

$$\omega^*(x, y) = u_x(x, y) + u_y(x, y)(n-1)q^*(x)$$

Theorem 1. *If (i) $u(\cdot,\cdot)$ is smooth in the relevant region $\mathcal{D}$, (ii) $y^*(\cdot)$ as defined above is strictly increasing above the ask price (in particular, there is a solution to the o.d.e. (4)), and (iii) in the relevant region, the sign of Miele's function is negative below*

[10] To make this statement rigorous, one has to impose additional technical assumptions on the function $u(\cdot,\cdot)$.

the graph of $y^*(\cdot)$ and positive above the graph of $y^*(\cdot)$, then $q^*(\cdot)$ forms a symmetric equilibrium and $y^*(\cdot)$ is the equilibrium supply function.

To intuitively understand the role of Miele's function, we recall a standard result from calculus that is usuful when the first order conditions fail. Consider an arbitrary continuous function that is piecewise differentiable, $f(\cdot)$. If we were to plot the sign of $f'(\cdot)$ in an interval $[a, b]$, and find an $x \in [a, b]$ such that in $[a, x)$, the sign of $f'(\cdot)$ is positive and in $(x, b]$ the sign is negative, then x maximizes $f(\cdot)$ in the interval. Even if x is at the corner of the interval or $f'(x)$ does not exist (i.e., the local conditions for optimality fail), x is the optimal. This result is a consequence of the fundamental theorem of calculus: for any $z \in [a, b]$, $f(x) - f(z) = \int_z^x f'(u)du > 0$.

In the proof of Theorem 1, we express the objective of a liquidity supplier as a line integral, where the goal is to choose the curve that maximizes the line integral along the curve. The difference between the payoff of two curves can be expressed as the line integral along a closed curve. Miele's function is analogous to the derivative of the line integral, and therefore we are interested in the sign of Miele's function. To verify optimality, we use Green's theorem, which is essentially a two dimensional version of the fundamental theorem of calculus: Green's theorem equates the line integral along a closed curve with the double integral of Miele's function in the region enclosed by the curve; i.e., the difference in payoffs between two curves that start at common point and end at a common point.[11]

Before we prove the theorem, we show how we apply the theorem in two examples.

The Exponential-Normal Example:

Glosten (1994) studies a special case of the exponential example, in which $\widetilde{I}$ and $\widetilde{s}$ are normally distributed with zero mean, the variance of $\widetilde{I}$ is $\alpha < 1$, the variance of $\widetilde{s}$ is $1 - \alpha$, and $\sigma = \gamma = 1$, so that $\widetilde{\theta}$ is a standard normal random variable. We have

$$v(\theta) = (1 - \alpha)\theta$$

and

$$v^+(\theta) = (1 - \alpha)\frac{\varphi(\theta)}{1 - \Phi(\theta)}$$

where $\varphi(\cdot)$ and $\Phi(\cdot)$ are the density function and the cumulative distribution function of a standard normal random variable, respectively. Thus, $u(x, y)$ is given by

$$u(x, y) = x(1 - \Phi(y + x)) - (1 - \alpha)\varphi(y + x).$$

[11] In the Appendix we prove a simple version of Green's theorem. The proof is an application of the fundamental theorem of calculus.

We set the parameters: $\alpha = 0.8$ and $X = 0.9$. Because $y \to u(X, y)$ changes signs only once, Y is the the root of the equation $u(0.9, y) = 0$. We solve and find that $Y = 3.387092$. Next we solve the o.d.e. (4) by integrating backward until the solution vanishes. Figure (1) shows the solution for different number of value traders. In particular, the supply function is monotone, as required in the theorem. Thus, to verify that we have found the equilibrium, we only need to check that the sign of $\omega^*(\cdot, \cdot)$ is positive above the graph of $y^*(\cdot)$ and negative below it. Figure (2) shows that this is indeed the case.

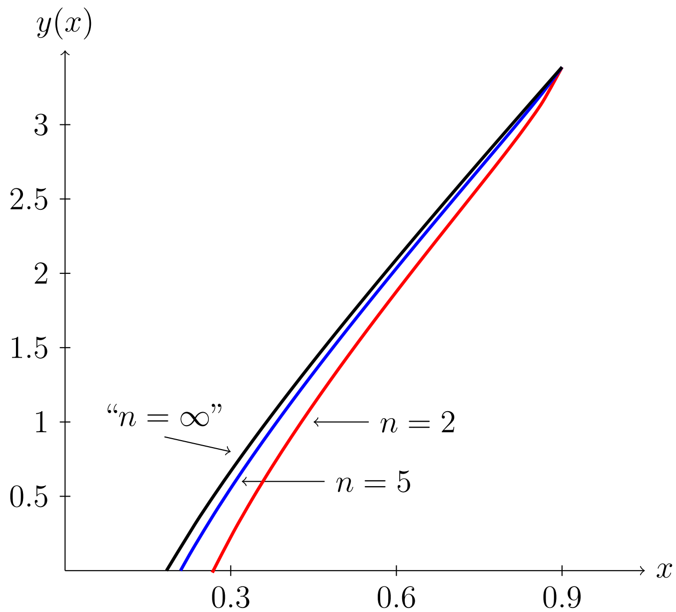

FIGURE 1. Equilibrium Supply Function in the Exponential-Normal Example. The figure shows the equilibrium supply function in the exponential normal case for different number of value traders. The parameters are $\alpha = 0.8$ and $X = 0.9$, and n is either $2, 5$ or the limiting case that satisfies the break even conditions ("$n = \infty$").

The Inelastic Demand Example:

Assume the active trader's orders size is uniformly distributed over the unit interval, and $v(\delta) = \delta$. The relevant region is the open rectangle $\{(x, y) : 0 \leq x, 0 \leq y \leq 1\}$ and

$$u(x, y) = \begin{cases} x(1-y) - 1/2 + y^2/2 & y \leq 1 \\ 0 & y > 1 \end{cases}$$

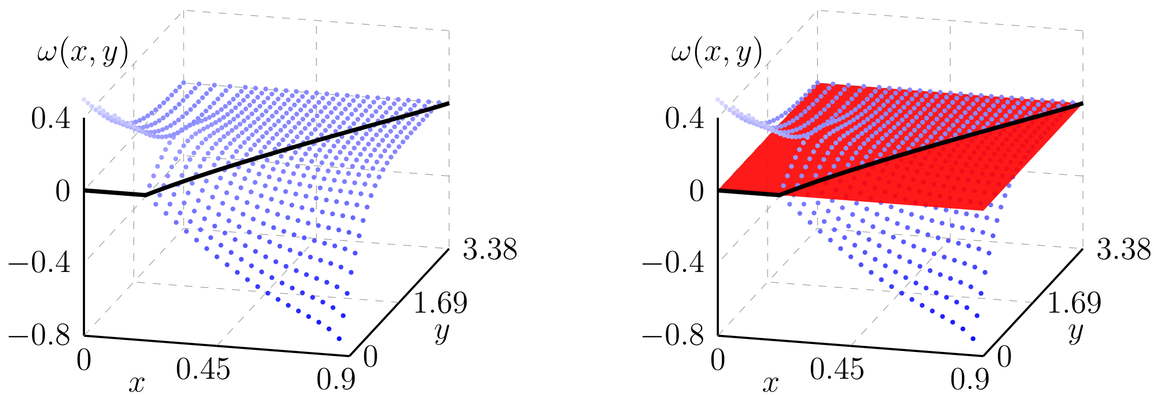

FIGURE 2. Miele's Function in the Exponential-Normal Example. The equilibrium supply function, $y^*(\cdot)$ is depicted n the plane $(x, y, 0)$. For better visualization, in the right panel we also show the surface $(x, y, 0)$. It can be seen that for every pair (x, y) above (below) the graph of $y^*(\cdot)$, $\operatorname{sgn}\omega^*(x, y)$ is positive (negative). The parameters are $\alpha = 0.8$ and $X = 0.9$ and $n = 5$.

Set the maximum price $X = 1$, and solve $u(1, y) = 0$ to find that $Y = 1$. The solution of the backward differential equation (4) is linear:

$$y(x) = x\frac{2n-1}{n-1} - \frac{n}{n-1}$$

The ask price is

$$ask_n = \frac{n}{2n-1}$$

hence the supply function is

$$y^*(x) = \begin{cases} 0, & 0 \leq x < \frac{n}{2n-1} \\ x\frac{2n-1}{n-1} - \frac{n}{n-1}, & \frac{n}{2n-1} \leq x \leq 1 \end{cases}$$

To verify that this is the equilibrium supply function, note that in the relevant region (i.e. $y \leq 1$) Miele's function is

$$\begin{aligned} \omega^*(x, y) &= \begin{cases} 1 - y, & 0 \leq x < ask_n \\ (1-y) - (y-x)\frac{(1-2n)}{n}, & ask_n \leq x \leq 1 \end{cases} \\ &= \begin{cases} 1 - y, & 0 \leq x < ask_n \\ \frac{n}{n-1}\left(y - y^*(x)\right), & ask_n \leq x \leq 1 \end{cases} \end{aligned}$$

Because for every price x below the ask price, $y(x) = 0$, we see that in $\mathcal{D}$, $\omega^*(x, y)$ is positive above the graph of $y^*(x)$ and negative below it. Thus, we have verified the equilibrium in this example.

Proof of Theorem 1: Assume the conditions in the theorem hold, and $n-1$ value traders use the strategy $q^*(\cdot)$. We need to show that $q^*(\cdot)$ is optimal for the n-th trader as well. Because $y^*(\cdot)$ is monotone, $q^*(\cdot)$ is a feasible strategy.

By inserting, in (3), the constraint into the objective, we remove $q(\cdot)$ from the objective and express the payoff in terms of $y(\cdot)$:

$$\int_0^X u(x, y(x))dy(x) - \int_0^X u(x, y(x))(n-1)q^*(x)dx$$

We note that the above is also the line integral of $u(x, y)dy - u(x, y)(n-1)q^*(x)dx$ along the graph of $y(\cdot)$. Thus, the payoff of every strategy $q(\cdot)$ equals the line integral along the graph of $y(\cdot)$, where $y(x) = \int_0^x q^*_{-i}(s) + q(s)ds$.

Let C denote a plane curve and let

$$I(C) = \int_C u(x, y)dy - u(x, y)(n-1)q^*(x)dx \tag{5}$$

We denote the graph of the function $y^*(\cdot)$ by C_{y^*}. To prove that $q^*(\cdot)$ forms a symmetric equilibrium, it is sufficient to show that $I(C_{y^*}) \geq I(C)$ for every plane curve that lies in relevant region $\mathcal{D}$, starts at the origin, and ends at a point on the vertical line $x = X$. The proof that it is sufficient to restrict attention to $\mathcal{D}$ is given in Lemma 1 in the Appendix.[12]

Consider a plane curve that lies above the graph of $y^*(\cdot)$, as shown in Figure (3). Let α consist of all those points that are above the graph of $y^*(\cdot)$ and below C. Let $\partial\alpha$ be the positively-oriented simple closed curve that forms the boundary of α;[13] i.e., $\partial\alpha$ is the union of C_{y^*}, the vertical plane curve that connects C_{y^*} to C, and $-C$.[14]

$$I(C_{y^*}) - I(C) \geq \oint_{\partial\alpha} u(x, y)dy - u(x, y)(n-1)q^*(x)dx \quad = \iint_\alpha \omega^*(x, y)dxdy \geq 0$$

[12] This problem is more general than the original problem: 1) We don't require that the graph be above the graph of the function $y^*(\cdot)(n-1)/n$. We ignore this condition because it was never binding is specific examples. 2) We don't require the arbitrary plane curve be the graph of a monotone function or a smooth function or even a continuous function.

[13] A simple closed curve is said to be positively oriented if, as we traverse the curve, the region enclosed by the curve is to our left. The arrows in Figure (3) indicate the directions.

[14] The minus sign before C indicates the reversal of orientation on C.

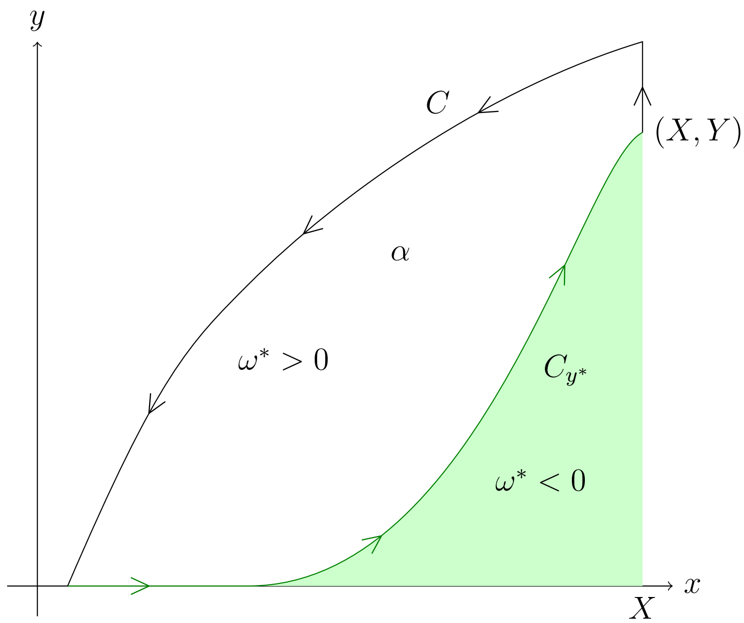

FIGURE 3. Green's Theorem and the optimality of C_{y^*}. The sign of $\omega^*(\cdot,\cdot)$ is negative in the shaded area and positive elswhere. The difference in the value of the line integral (5) along C_{y^*} and the line integral along C is greater than the double integral of $\omega^*(\cdot,\cdot)$ in the region α. Because $\omega^*(\cdot,\cdot)$ is positive in α, the double integral is positive.

where the first inequality is from the definition of Y.[15] The second equality is Green's Theorem,[16] and the final inequality is because the sign of $\omega^*(\cdot,\cdot)$ is positive in α.

We now consider a plane curve C as depicted in Figure (4). Let α be the set of points that are above C_{y^*} and below C, and let β be the set of points that are below C_{y^*} and above C. Let $\partial\alpha$ and $\partial\beta$ be the positively-oriented simple closed curves that forms

[15]The definition of Y implies that the line integral (5) is negative for any vertical curve that starts at (X, Y). Indeed, the a vertical plane curve, C, starts at (X, Y) and ends in (X, Y'). Because "$dx = 0$," we have $I(C) = \int_{Y}^{Y'} u(X, y)dy \leq 0$.

[16]In Appendix B, we prove that Green's Theorem can be applied even though the integrand is not smooth: $q^*(\cdot)$ has a point of discontinuity at $x = ask$, and $\omega^*(\cdot,\cdot)$ has a vertical curve of discontinuity at $x = ask$.

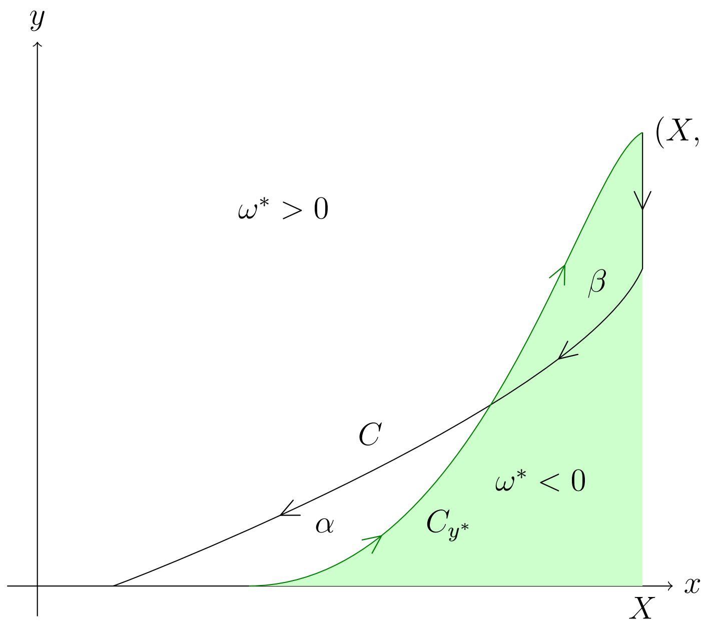

FIGURE 4. Green's Theorem and the optimality of C_{y^*}. By Green's Theorem and the definition of Y, $I(C_{y^*}) - I(C)$ greater than the double integral of ω^* in the region α minus the double integral of $\omega^*(\cdot,\cdot)$ in the region β. Because $\omega^*(\cdot,\cdot)$ is positive in α and negative in β, it follows that the difference $I(C_{y^*}) - I(C)$ is positive.

the boundary of α and β, respectively. We have

$$\begin{aligned} I(C_{y^*}) - I(C) &\geq \oint_{\partial\alpha} u(x,y)dy - u(x,y)(n-1)q^*(x)dx \\ &\quad - \oint_{\partial\beta} u(x,y)dy - u(x,y)(n-1)q^*(x)dx \\ &= \iint_{\alpha} \omega^*(x,y)dxdy - \iint_{\beta} \omega^*(x,y)dxdy \geq 0 \end{aligned}$$

where the first inequality is from the definition of Y.[17] Note that we put a minus sign before the line integral along $\partial\beta$ because we traverse $\partial\beta$ is the negative direction.

In a similar way we show that the payoff associated with $y^*(\cdot)$ (and hence with $q^*(\cdot)$) is greater than the payoff associated with all other strategies. This completes the proof of Theorem (1)

Note that the dynamic programming approach and Miele's approch agree, as expected: Miele's function is zero along the graph of $y^*(\cdot)$. Whereas the Bellman

[17]See Footnote 15

Equation did not give us information off the path, examining the sign of Miele's function off the path tells us whether the extremum we have identified is a minimum or a maximum of the objective.

Finally, note that if the solution to the o.d.e. (4) is non monotone, then the optimal strategy does not have a meaning in a pure limit order book. However, if we were to model a dealers market where larger orders may get better prices, then the cumulative number of shares offered may be a decreasing function.[18]

6. An Artificial Market game

The key condition in the verification theorem is the sign of Miele's function. We can find economic environments in which the o.d.e. (4) does not define the equilibrium supply function. In the following example we choose $u(\cdot,\cdot)$ in a manner that there are two functions $y : [ask_n, X] \to R$ that satisfy the equation $\omega^*(x, y(x)) = 0$.[19]

Consider the exponential example again, only that this time assume that θ is a standard uniform random variable, $v(\theta) = \alpha\theta^2$, and $\gamma = \sigma = 1$. This implies that

$$u_y = -x + \alpha(x + y)^2$$
$$u_x = 1 - x - y + u_y$$

Consider now an arbitrary maximum price X such that the solution to the o.d.e. (4) exists. While I cannot solve analytically the o.d.e., we can still see that the function $y \to w^*(x, y)$ is quadratic in the interval $[ask_n, X]$. Moreover, because $y^*(ask_n) = 0$, at least in a small interval $(ask_n, ask + \epsilon)$, $y \to \omega^*(x, y)$ is convex.

Because we don't know $y^*(x)$, we cannot express the two zeros of $y \to w^*(x, y)$ in terms of x. However, we can express them in terms of x and the unknown $y^*(x)$. The first root, by construction, is $y^*(x)$, and the second root is

$$r(x, y^*(x)) = \frac{-2x\alpha + \alpha x^2 + x + \alpha x y^*(x) - \alpha y^*(x)}{\alpha(1 - x - y^*(x))}$$

If we take $\alpha < 1/2$, we have $r(x, y^*(x)) > y^*(x)$. Thus, in the interval $(ask_n, ask + \epsilon)$, $w^*(x, y) > 0$ "below" the smaller root (and negative "between" the two roots). Therefore, we have proved that there is a region where the sign of Miele's function is positive below the graph of $y^*(\cdot)$ (and negative above it). Figure 5 shows an example

[18] A dealer can offer discounts for large orders.

[19] By the definition of $\omega^*(\cdot,\cdot)$, $y^*(\cdot)$ always satisfies the equation.

where we solve the o.d.e. (4) numerically. The example illustrates how easily we can construct a profitable deviation.

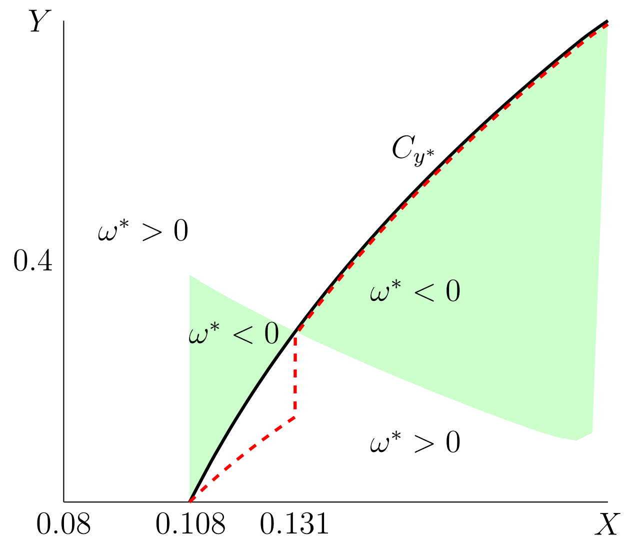

FIGURE 5. A Profitable Deviation when Miele's function is positive below the graph of y^*. This figures shows a special case of the exponential example, where θ is a standard uniform random variable and $v(\theta) = \alpha\theta^2$. The parameters are $n = 2$, $\alpha = 0.2$ and $X = 0.19999$ (which implies $Y = 0.79599$). The region where the sign of $\omega^*(\cdot,\cdot)$ is negative is shaded. The dotted graph illustrate a profitable deviation. The profitable deviation is not to offer any share up to the price 0.131, and at that price to put a discrete offer with a size of $\int_0^{0.131} q^*(x)dx$ shares. For prices greater than 0.13, the deviation is identical to $q^*(\cdot)$.

Whether or not the solution to the o.d.e. (4) defines an equilibrium supply function, the solution converges to the supply function that (at least in regular economies) satisfies the break even conditions. This leads us to consider the following artificial game.

Given n liquidity suppliers, let ask_n be as before; i.e., ask_n is the price at which the solution to the backward differential equation (4) vanishes. Consider the artificial problem of the i-th liquidity supplier:

$$\begin{cases} \min\limits_{q(x)\geq 0} \int_{ask_n}^{X} u(x,y) q_i(x) dx \\ \text{subject to} \\ dy(x) = q_{-i}(x) + q(x)dx \\ y(ask_n) = 0 \\ \int_{ask_n}^{X} q_i(x)dx = Y/n \end{cases}$$

An equilibrium of the artificial game is an n-tupple of functions $(q_1(\cdot), q_2(\cdot), \ldots, q_n(\cdot))$ such that for each i, $q_i(\cdot)$ solves the artificial problem. In this artificial game, each liquidity supplier has to offer a total of Y/n shares on the interval of prices $[ask_n, X]$ in a manner that minimizes gains.

Theorem 2. *Let $y^*(\cdot)$, $w^*(\cdot)$ and $q^*(\cdot)$ be as defined in the previous section. If, for all $x \in [ask_n, X]$, the sign of $\omega^*(\cdot, \cdot)$ is positive below the graph of $y^*(\cdot)$ and negative above it, then $q^*(\cdot)$ forms a symmetric equilibrium of the artificial game and $y^*(\cdot)$ is the equilibrium supply function of the artificial game*

The proof is similar to the proof of Theorem 2, so we only provide a sketch. We take $q_{-i}(\cdot) = (n-1)q^*(\cdot)$. We then write the objective of the artificial problem as a line integral and look for the curve that minimizes the objective in the class of curves that start at $(ask_n, 0)$ and end at $(X, Y/N)$. Using Green's theorem, we show that the graph of $y^*(\cdot)$ solves the minimization problem.

The Uniform Example, Continuted:

Let X be an arbitrary maximum price such that the backward equation (4) admits a solution. Miele's function is given by

$$\omega^*(x, y) = u_x(x, y) - u_y(x, y)\frac{u_x(x, y^*(x))}{u_y(x, y^*(x))}$$

In the uniform example,

$$u_y = -\mu x/(2L), \quad u_x = \frac{L - x - \mu(x + y)}{2L}$$

Hence,

$$\omega^*(x, y) = u_x(x, y) - u_x(x, y^*(x)) = -\frac{\mu}{2L}(y - y^*(x))$$

Thus, $\omega^*(x, y) > 0$ if and only if $y < y^*(x)$, and we conclude that $y^*(\cdot)$ is the equilibrium supply function of the artificial game.

To see a specific example, we take the parameters $L = 1$, $\mu = 1/2$, and $X = 1/\sqrt{3}$. The supply function that satisfies the break even book is

$$y_\infty(x) = \begin{cases} 0 & x < 1/3 \\ 2 - \frac{3}{2}x - \frac{1}{2x} & 1/3 \le x \le \frac{1}{\sqrt{3}} \end{cases}$$

and we can verify that the environment is regular: below the graph of $y_\infty(x)$, the sign of $u(x,y)$ is positive and above it the sign is negative. This book is what Glosten (1994) reports in page 1144.[20]

We find that $Y = 2-\sqrt{3}$, and we solve the o.d.e. (4) backward and find the equilibrium supply function of the artificial game. We cannot solve for the ask price analytically, but we can easily do it numerically. For example, for $n=2$, the ask price is 0.4007035757. Above the ask price (for arbitrary n), the supply function is

$$y^*(x) = 2 - \frac{3nx}{2n-1} - \frac{(n-1)}{(2n-1)} x^{-n/(n-1)} 3^{-1/(2n-2)}$$

Note that this economic environment is regular, the equilibrium supply function is monotone, and if we use a subscript n to denote the number of liquidity suppliers in the model, then $y^*_n(x) \leq y^*_{n+1}(x)$ and the limit is $y_\infty(x)$. This is in accordance with our discussion that followed (4).

This means that in the equilibrium of the artificial game, liquidity suppliers don't expect to lose. The constraint on the number of shares together with the price interval is such that, in equilibrium, all offers are profitable. This ends the example.

Corollary 1. *If the economic environment is such that the profitability of a marginal offer satisfies: (i)*

$$u_y < 0 \text{ or } u_x > 0$$

and (ii)

$$\frac{\partial}{\partial y}\frac{u_x(x,y)}{u_y(x,y)} \geq 0$$

then the solution to the o.d.e. (4) *is the equilibrium supply function of the artificial market game.*

The conditions imply that $y \to \omega^*(x,y)$ is a decreasing function. By construction, $\omega^*(x, y^*(x)) = 0$, and hence the result follows from Theorem 2.

The Inelastic Example, Continuted:

In the inelastic example, we always have $u_x > 0$. Thus, if $v(y)$ and $F(y)$ are such that

$$\frac{\partial}{\partial y}\frac{u_x(x,y)}{u_y(x,y)} \geq 0$$

[20]In our analysis, the marginal price x is the state variable, while in Glosten (1994) the total number of shares, denoted there by δ, is the state variable. Replace δ with $y(x)$ and x with $R'(\delta)$ to see that the competitive books are identical.

then the solution to the o.d.e. (4) defines an equilibrium supply function of the artificial market game.

7. Conclusion

In this paper we reassessed the break even conditions. We developed a model of imperfect competition in liquidity provision and used the Bellman equation to identify a candidate to be the equilibrium book. The candidate converges, as the number of liquidity suppliers increases, to a limit order book that satisfies the break even conditions.

In some economic environments, we can verify that the candidate is indeed an equilibrium. However, in other economic environments it is the equilibrium of an artificial game where liquidity suppliers post offers in a manner that minimizes gains. Thus, our results demonstrate that the intuition that underlies the break even conditions is incomplete.

Appendix A

Lemma 1. *For every supply function* $y : [0, X] \to R$ *with* $y(0) = 0$*, there is a function* $y_\epsilon : [0, X] \to R$ *with a graph that lies in* $\mathcal{D}$*, starts at the origin, and ends at a point on the vertical curve* $x = X$ *such the line integral* (5) *has the same value; i.e.,* $I(y(\cdot)) = I(y_\epsilon(\cdot))$.

If the graph of $y(\cdot)$ lies in $\mathcal{D}$, then we take $y_\epsilon(\cdot) = y(\cdot)$. Say now that the supply function has a graph that, at least part of, lies outside $\mathcal{D}$.

Because of the price priority rule, $\epsilon_y \leq 0$, where $\epsilon(x, y)$ is the probability of execution of a marginal offer at (x, y). This implies that the region $\mathcal{D}$ is simple (i.e., it has no holes), and if $(x, y(x)) \notin \mathcal{D}$ then the point (x, y) is "above" the relevant region $\mathcal{D}$. Recall that outside the relevant region, the probability of execution is zero and hence the marginal profit is zero.

Define

$$y_\epsilon(x) = \begin{cases} y(x) & (x, y(x)) \in \mathcal{D} \\ \max_y\{y : (x, y) \in \mathcal{D}\} & \text{Otherwise} \end{cases}$$

It follows that $u(x, y_\epsilon(x) = u(x, y(x))$ for all $x \in [0, X]$. Thus, the line integral along the graphs of both functions is identical. This completes the proof.

Appendix B. Green's Theorem

Green's theorem (or Green's identity) relates a double integral over a region with a line integral over the boundary of the region:

$$\iint\limits_{\alpha} Q_x - P_y dA = \oint_{\partial\alpha} Qdy + Pdx \tag{6}$$

where α is a simple closed region, $\partial\alpha$ is the positively oriented closed curve that forms its boundary, and Q and P are functions of x and y with continuous partial derivative in the region as well as on the boundary.[21] In our model, $P(x, y) = -u(x, y)(n-1)q^*(x)$, and we want to use Green's identity (6) also when $q(\cdot)$ has a point of discontinuity.[22] Thus, neither P nor its partial derivative is continuous. However, the curve of discontinuity of P and P_y is vertical, and we will see that in that case Green's theorem still holds

To keep this paper self-contained, we prove the Green's Theorem under the assumption that the region α is both x-simple and y-simple.[23] We then show how to prove the theorem when $q^*(\cdot)$ is piecewise continuous.

A region α in the xy-plane is x-simple if there are two functions $k(y)$ and $l(y)$ such that

$$\alpha \{(x, y) : k(y) \leq x \leq l(y), c \leq y \leq d\}$$

A region α in the xy-plane is y-simple if there are two functions $f(x)$ and $h(x)$ such that

$$\alpha = \{(x, y) : a \leq x \leq b, f(x) \leq y \leq h(x)\}$$

Green's theorem is in fact the sum of two identities, (7) and (8) below, and we prove each identity separately. We start with

$$\oint_{\partial\alpha} Qdy = \iint\limits_{\alpha} Q_x dA \tag{7}$$

[21]The definition of the orientation of a closed curve is provided in footnote 13.

[22]More specifically, $q^*(\cdot)$ has a point of discontinuity at the ask price.

[23]For example, the regions bounded between the graphs of two monotone functions are both x-simple and y-simple.

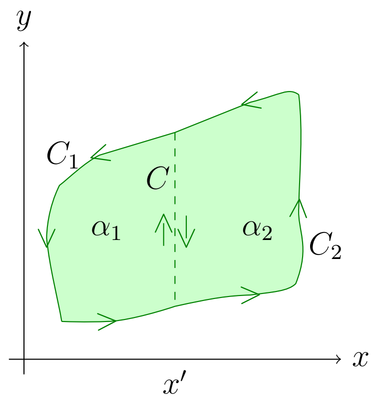

FIGURE 6. Green's Theorem with a Discontinuous Integrand. This figure illustrates that if the curve of discontinuity of the integrand P is vertical, then the Green's identity still holds because the line integral of P along the the curve C is zero.

Because α is x-simple, we have

$$\oint_{\partial\alpha} Qdy = \int_c^d Q(k(y),y)dy + \int_d^c Q(l(y),y)dy$$
$$= \int_c^d \int_{l(y)}^{k(y)} Q_x(x,y)dxdy = \iint\limits_{\alpha} Q_x(x,y)dA$$

Next, we show

$$\oint_{\partial\alpha} Pdx = \iint\limits_{\alpha} -P_y dA \tag{8}$$

Because α is also y-simple, we have

$$\oint_{\partial\alpha} Pdx = \int_a^b P(x,f(x))dx + \int_b^a P(x,h(x))dx$$
$$= -\int_a^b \int_{f(x)}^{h(x)} P_y dydx = \iint\limits_{\alpha} -P_y dA$$

This completes the proof of the Green's Theorem when the region is both x-simple and y-simple.

Suppose now that P has a curve of discontinuity, call it C, such that C splits $\partial\alpha$ into C_1 and C_2 and splits the region α into two simple regions α_1 and α_2 (see Figure (6)). The function P and its partial derivative do not show in identity (7), so to prove Green's theorem, we only need to consider the identity (8) .

The line integral and the double integral in (8) are independent of how we define P and P_y along C, but in general the identity (8) does not hold. To see this, we use Green's theorem in the left region (see Figure (6)):

$$\int_{C_1} P dx + \iint_{\alpha_1} P_y dA = -\int_C P^- dx$$

where P^- is the left limit of P along its curve of discontinuity, C. Similarly, we use Green's theorem in the right region (note that now we traverse C in the opposite direction, see Figure (6)) :

$$\int_{C_2} P dx + \iint_{\alpha_2} P_y dA = \int_C P^+ dx$$

where P^+ is the right limit of P along C. We add the two and get

$$\oint_{\partial\alpha} P dx + \iint_{\alpha} P_y dA = \int_C P^+ dx - \int_C P^- dx$$

However, when the curve of discontinuity is vertical, then along C "$dx = 0$." Thus, $\int_C P^+ dx = 0$ and $\int_C P^- dx = 0$. We conclude that the Green's theorem holds even when P has a vertical curve of discontinuity.

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