# A Bayesian Approach to Real Options: The Case of Distinguishing Between Temporary and Permanent Shocks*

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### Abstract

This paper studies the optimal timing of investment in the presence of uncertainty about both future and past shocks. Specifically, in addition to the standard Brownian uncertainty driving traditional real options models, we also allow for Bayesian uncertainty over distinguishing between the temporary or permanent nature of past cash flow shocks. As a result, the evolving uncertainty is no longer constant, and is driven by Bayesian updating. We solve for the optimal investment rule and show that the implied investment behavior differs significantly from that predicted by standard real options models. For example, in contrast to the standard real options implications, firms hold both a traditional Brownian "option to wait" as well as a Bayesian "option to learn", investment may occur at a time of stable or decreasing cash flows, investment may respond sluggishly to positive cash flow shocks, and investment timing will critically depend on the maturity structure of the project cash flows.

*Keywords:* irreversible investment, real options, Bayesian updating, learning, temporary and permanent shocks, mean reversion

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# 1 Introduction

During the past two decades the real options approach to valuation of irreversible investment opportunities has become part of the mainstream literature in financial economics. The central idea is that the opportunity to invest is equivalent to an American call option on the underlying investment project. As a consequence, the problem of optimal investment timing is analogous to the optimal exercise decision for an American option. Applications of the real options approach are now numerous.[1]

One feature which is common to virtually all real options models is that the underlying uncertainty is about future shocks only. Since the future value of the asset is uncertain, there is an important opportunity cost of investing today: the value of the asset might go up so that tomorrow will be an even better time to invest. This opportunity cost is often referred to in the literature as the "option to wait." The implied investment strategy in this setting is to wait until the value of the asset reaches some upper threshold value and then invest.

In this paper we propose a very different kind of real options problem. While uncertainty about future shocks is important, it may not be the most important, let alone unique uncertainty faced by a firm. Indeed, another critical source of uncertainty is uncertainty about *past* shocks. This is the case when the firm observes the past shock, but fails to identify its exact properties. As time passes, the firm updates its beliefs about the past shock. Therefore, unlike the standard models, the evolving uncertainty is driven by both the standard Brownian uncertainty over future cash flows as well as by Bayesian updating, or learning about the nature of past cash flow shocks. This gives rise to an additional trade-off between investing now and waiting, embodied by what we call the "option to learn."

Specifically, this paper focuses on the case where uncertainty about past shocks comes from the firm's inability to distinguish between temporary and permanent shocks to the cash flow process. To gain intuition underlying the trade-off between investing now and learning, consider a positive shock to cash flows. In the standard setting the firm's strategy would be rather simple: the firm should invest if and only if the shock is high enough so that the value of the cash flow process exceeds some threshold level. However, when the firm does not know if the shock is temporary or permanent, it may want to wait in order to learn more about the identity of the shock. Indeed, if the firm waits until tomorrow

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[1] The early literature is well summarized in Dixit and Pindyck (1994). McDonald and Siegel (1986) provided the (now standard) setting, that was later extended to account for a time-to-build feature (Majd and Pindyck (1987)) and strategic interactions among several option holders (Grenadier (1996, 2002), Lambrecht and Perraudin (2003), Novy-Marx (2007)). Real options modeling is used to study specific industries such as natural resources (Brennan and Schwartz (1985)) or real estate (Titman (1985) and Williams (1991)). Recent developments include incorporating agency conflicts (Grenadier and Wang (2005)) and behavioral preferences (Grenadier and Wang (2007), Nishimura and Ozaki (2007)) into the standard setting.

and the value of the cash flow process is still at the high level, then the past shock is more likely to represent a positive fundamental change. Similarly, if the cash flow process decreases, then there is a greater chance that the past positive shock was simply a result of temporary, non-fundamental fluctuations.

Our argument is based on two building blocks that distinguish this model from the real options literature. The first building block is the presence of both temporary and permanent shocks to cash flows. While virtually all real options models focus solely on permanent shocks, the presence of temporary shocks is a natural feature of many real-world economic environments. The focus on only permanent shocks is clearly a simplification that permits one to model the cash flow process as a geometric Brownian motion. However, as Gorbenko and Strebulaev (2008) show, assuming a geometric Brownian motion cash flow process leads to a number of undesirable empirical properties.[2]

The second building block is the inability of the firm to distinguish between permanent and temporary shocks. This is an especially important feature of investments in natural resources, where it is often unclear when a change in the commodity price represents a fundamental or ephemeral shift. Indeed, as the World Bank's Global Economic Prospects annual report for the year 2000 states:

> "Most important, distinguishing between temporary and permanent shocks to commodity prices can be extraordinary difficult. The swings in commodity prices can be too large and uncertain to ascertain their causes and nature. The degree of uncertainty about duration of a price shock varies. For example, market participants could see that the sharp jump in coffee prices caused by the Brazilian frost of 1994 was likely to be reversed, assuming a return to more normal weather. By contrast, most analysts assumed that the high oil prices during the mid-1970s and early 1980s would last indefinitely."[3]

We show that by augmenting the traditional Brownian uncertainty of standard real options models with Bayesian uncertainty, a number of very important novel implications emerge. First, in contrast to the standard real options setting in which the investment is triggered when cash flows rise to a constant trigger, we find that once we account for Bayesian uncertainty about past shocks, the investment trigger is a function of the firm's changing beliefs about the nature of past shocks. This distinction is due to the fact that while in standard real options models Brownian uncertainty is constant over time, in our Bayesian framework uncertainty depends on the timing of past shock. More specifically, uncertainty is high soon after the arrival of a large shock since the firm is very unsure whether it is permanent or temporary. However, as time after the shock goes by, uncertainty declines as the firm becomes progressively more confident in a past shock's

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[2] For example, it implies that the volatilities of the cash flow and asset growth are equal, while empirically volatilities of cash flow growth are much higher than volatilities of asset value growth.

[3] World Bank, Global Economic Prospects and Developing Countries 2000, p. 110.

permanence. Thus, in addition to the standard option to wait for realizations of future shocks, we now also have a valuable option to wait in order learn more about the nature of past shocks. We find that these two options to delay investment are additive, and appear to be of similar magnitudes when past shocks are significant. Ignoring either option would appear to lead to a similar investment timing error. The Bayesian learning feature thus implies a sluggish response of investment to positive cash flow shocks. Intuitively, a positive shock increases uncertainty since the firm is not sure if it is permanent or temporary. As a result, the option to learn about the nature of past shocks becomes valuable, so the firm may choose to respond sluggishly.

A second important implication of the Bayesian model is that investment may occur in the face of stable or even decreasing cash flows. Since standard real options models imply that investment will be triggered when shocks push the underlying cash flow level up to a fixed upper threshold, the firm can rationally exercise the investment option only when the underlying cash flows increase. However, when the traditional real options model is extended to include the Bayesian feature, the firm may invest even when the cash flow process is stable or declining simply as it becomes more certain about the permanent nature of the past shocks. Because of this, accounting for uncertainty about past shocks may lead to a failure of the "record-setting news principle." The record-setting news principle is that investment occurs only at instants in which the value of the cash flow process is the highest in its whole history.[4] While the record-setting news principle holds for a large class of real options models (Boyarchenko (2004)), the addition of Bayesian uncertainty over the nature of past shocks can overturn this result. Thus, even when current cash flows are not at their all-time maximum, the decline in uncertainty about past shocks can trigger investment. While violation of the record-setting news principle and investment in the face of stable or decreasing cash flows can be obtained in other models with more than one state variable, our argument is special since it links these features to the timing of observable shocks and thereby offers an intuitive explanation for firms (and industries) investing in markets where cash flows are stable, or even declining. For example, as we discuss below, learning about the persistence of the 1973 oil price shock was one likely reason for the rapid office construction in Denver and Houston during the late 1970's and early 1980's while the underlying property cash flows were declining.

Finally, in the context of valuations that are driven by the possibility of both temporary and permanent shocks, the timing of cash flows can be quite important. This is in contrast to standard real options models where for a given net present value the timing of the cash flows is irrelevant to the investment timing decision. Specifically, we show that the lesser the "front-loadedness" of the project's cash flows, the later the investment will take place. Intuitively, the importance of the nature of past shocks decreases in the front-loadedness of the project cash flows. For example, consider the investment project

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[4] See Boyarchenko (2004) for a discussion of the record-setting news principle, a term credited to Maxwell Stinchcombe.

that consists of creating an asset at some fixed construction cost and then re-selling the project at a market price. This is the case when the project is extremely front-loaded. Clearly, since the firm gets all cash flows from the project immediately at the chosen time, the nature of the past shocks has no effect on the firm's optimal investment strategy. On the other hand, when the firm's investment pays off with a flow of cash flows over a long period of time, such as the development of an oil well, cash flows are relatively back-loaded. In this case learning if a past increase in oil prices was a fundamental shift or simply a temporary fluctuation has important option value for the firm.

Several papers deserve mentioning as being the most closely related to our model. Gorbenko and Strebulaev (2008) incorporate temporary shocks into a contingent claims framework of capital structure and show that the presence of temporary shocks provides a potential explanation of several puzzles in corporate finance. Their paper differs from ours in two important respects. First, and most importantly, our argument heavily relies on the inability of the firm to distinguish between temporary and permanent shocks, while Gorbenko and Strebulaev (2008) do not have any learning in their model. Second, while we consider firms' investment decisions, their focus is on financing policy. Our paper is also related to Decamps et al. (2005) who solve for the optimal investment timing when the firm is uncertain about the drift of the state process. There are two major differences between their paper and ours. First, while in our paper the firm learns about the nature of past shocks to the state process, in Decamps et al. (2005) the firm learns about the parameters of the state process itself. Second, in Decamps et al. (2005) the timing and degree of uncertainty is pre-specified, while in our model uncertainty is proportional to past cash flow shocks whose timing is also uncertain to the firm. Another related paper is Miao and Wang (2007) who solve for the optimal entrepreneur's decision to exit a business in the presence of idiosyncratic nondiversifiable risk. Specifically, Miao and Wang (2007) consider an entrepreneur with incomplete information about his entrepreneurial abilities who chooses between continuing entrepreneurial activity and taking a safe job. Our paper is distinguished from theirs in that our focus is on cash flow uncertainty, as well as our model's inclusion of both Brownian and Bayesian uncertainty, as well as a more general Bayesian updating process.[5] Finally, our paper also shares the learning feature with Lambrecht and Perraudin (2003) who study competition between two firms for a single investment opportunity when information about investment costs is private. Because of this, as time goes by, and the competitor has not invested yet, each firm updates its belief about the competitor's investment costs upward.[6]

The remainder of the paper is organized as follows. Section 2 provides a set-up of the

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[5] See also other papers on optimal experementation that deal with problems of optimal control under learning (e.g., Jovanovic (1979) Bolton and Harris (1999), Keller and Rady (1999), Moscarini and Smith (2001)).

[6] Our paper is also related to Moore and Schaller (2002), who extend the neoclassical q theory of investment by allowing for permanent and temporary shocks to interest rates, and uncertainty about them.

model, and for purposes of providing intuition solves the most simple case. Section 3 focusing on analyzing the interaction between the two fundamental sources of cash flow uncertainty: Brownian and Bayesian. Section 4 provides a discussion of some interesting implications of the model. Section 5 focuses on the impact of having multiple shocks, allowing for a very general Bayesian updating process. Finally, Section 6 concludes.

# 2 Model Set-Up and a Simple Case

In this section we describe the set-up of the model which incorporates the essential features of investment timing into a Bayesian framework. Then, to show the main intuition behind the argument, we consider the simplest possible setting as a special case. In subsequent sections we solve for the more general cases.

## 2.1 The Investment Option

Consider a standard real option framework in which a firm contemplates irreversible investment. By paying the investment cost $I$, the firm obtains the perpetual cash flow $X(t)$. The firm is free to invest in the project at any time it so chooses. The firm has a constant discount rate $r > 0$.

The evolution of the cash flow process consists of two components. As usual in the literature, the cash flow process evolves according to the geometric Brownian motion with drift $\alpha$ and diffusion $\sigma$, where we assume $\alpha < r$.[7] In addition to this, the cash flow process is subject to jump shocks, which can be temporary or permanent. Upon the arrival of a shock, the cash flow process jumps from $X(t)$ to $X(t)(1+\varphi)$, with $\varphi > 0$.[8] A permanent shock changes the value of the cash flow process forever, while a temporary shock changes the value of the cash flow process only temporarily until it reverts at some random point in the future. Importantly, the firm is unable to distinguish between permanent and temporary shocks, and uses Bayesian updating to assess their relative likelihoods. We assume that the arrival and reversal (of the temporary shock) follow independent Poisson jump processes. Specifically, we assume that the permanent jump arrives with intensity $\lambda_1$, the temporary jump arrives with intensity $\lambda_2$, and an existing temporary shock reverses with intensity $\lambda_3 > \lambda_2$.

Mathematically, the evolution of the cash flow process is given by

$$dX(t) = \alpha X(t)\,dt + \sigma X(t)\,dB(t) + \varphi X(t)\,dM(t) - \frac{\varphi X(t)}{1+\varphi} dN_k(t)\,,\ X(0) = X_0 \quad (1)$$

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[7] To be more precise, the upper bound on $\alpha$ that guarantees finite values depends on the particular specification of the model. In the model of Section 3 and the simple case of Section 2, we assume $\alpha < r$. In the model of Section 5, we assume that $\alpha + \lambda_1\varphi < r$, where $\lambda_1$ and $\varphi$ are defined below.

[8] The model can be extended to the case of stochastic jump amplitudes, albeit at the cost of additional mathematical complexity. However, the basic intuition and results remain unchanged.

where $dB(t)$ is the increment of a standard Wiener process, and $dM(t)$ and $dN_k(t)$ are independent Poisson processes corresponding to the arrival and reversal of jumps, respectively. Since at any instant a new permanent shock arrives with intensity $\lambda_1$ and a new temporary shock arrives with intensity $\lambda_2$, the intensity of $dM(t)$ equals $\lambda_1 + \lambda_2$. Since only temporary shocks revert, the intensity of $dN_k(t)$ equals $k\lambda_3$, where $k$ is the number of outstanding temporary shocks.

## 2.2 The Bayesian Learning Process

Since the firm is unsure whether past shocks are permanent or temporary, an important state variable is the firm's belief about the identity of past shocks. Consider moments $t$ and $t + dt$ for any $t$ and infinitesimal positive $dt$. Depending on the history between $t$ and $t + dt$, there are three cases to be analyzed[9]:

1. no new shocks occur or outstanding shocks reverse between $t$ and $t + dt$;
2. a new shock occurs;
3. an outstanding shock reverses.

Consider the first case where there are no changes in the number of outstanding shocks between $t$ and $t + dt$. Suppose that there are $n$ outstanding shocks and at time $t$ the firm assesses the probability that $k$ of them are temporary at $p_k(t)$, $k = 1, ..., n$. If $k$ of the outstanding shocks are temporary, over a short period of time $dt$ a shock reverts back with probability $k\lambda_3 dt$. Similarly, a new shock arrives with probability $(\lambda_1 + \lambda_2)\, dt$. Thus, conditional on $k$ of the shocks being temporary, the probability that there are no new shocks or reversions over $dt$ equals $1 - (\lambda_1 + \lambda_2)\, dt - k\lambda_3 dt$. Using Bayes rule, the posterior probability $p_k(t + dt)$ is given by

$$p_k(t + dt) = \frac{p_k(t)\left(1 - (\lambda_1 + \lambda_2)\, dt - k\lambda_3 dt\right)}{1 - (\lambda_1 + \lambda_2)\, dt - \lambda_3 \sum_{i=1}^{n} p_i(t)\, i dt}, \quad k = 1, ..., n. \tag{2}$$

Eq. (2) is a direct application of Bayes rule. The numerator is the joint probability of having $k$ temporary outstanding jumps and observing no jumps or reversions between $t$ and $t + dt$. The denominator is the sum over $i = 0, 1, ...$ of joint probabilities of having $i$ temporary outstanding jumps and observing no jumps or reversions between $t$ and $t + dt$.

We can rewrite (2) as

$$\frac{p_k(t + dt) - p_k(t)}{dt} = -\frac{\lambda_3 p_k(t)\left(k - \sum_{i=1}^{n} p_i(t)\, i\right)}{1 - (\lambda_1 + \lambda_2)\, dt - \lambda_3 \sum_{i=1}^{n} p_i(t)\, i dt}. \tag{3}$$

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[9] Note that the probability of observing more than one new shock or reversion between $t$ and $t + dt$ has the order $(dt)^2$. Since $dt$ is infinitesimal, we can ignore these cases. The same is true for the likelihood of both a new shock and a reversal occurring at the same instant.

Taking the limit as $dt \to 0$,

$$\frac{dp_k(t)}{dt} = -\lambda_3 p_k(t)\left(k - \sum_{i=1}^{n} p_i(t)\, i\right). \tag{4}$$

The dynamics of $p_k(t)$ while there are no new shocks or reversions is intuitive and has two interesting properties. First, $p_k(t)$ increases in time when $k < \sum_{i=1}^{n} p_i(t)\, i$, and decreases in time, otherwise. Intuitively, when $k < \sum_{i=1}^{n} p_i(t)\, i$, the likelihood of reversion conditional on having $k$ outstanding temporary jumps is lower than the unconditional likelihood of reversion. Therefore, when the firm does not observe a jump or reversion, it updates its beliefs $p_k(t)$ upward. The opposite is true when $k > \sum_{i=1}^{n} p_i(t)\, i$. Second, the speed of learning is proportional to $\lambda_3$. In other words, if temporary shocks are more short-term, then the firm updates its beliefs faster than if they are more long-term.

Now, consider the second case. If a new shock occurs between $t$ and $t + dt$, then the updated beliefs equal

$$p_k(t + dt) = p_{k-1}(t)\frac{\lambda_2}{\lambda_1 + \lambda_2} + p_k(t)\frac{\lambda_1}{\lambda_1 + \lambda_2}. \tag{5}$$

The intuition behind (5) is relatively simple. When a new shock occurs, it can be either permanent or temporary. After the shock, there can be $k$ temporary shocks outstanding either if there were $k - 1$ temporary shocks and the new shock is temporary or if there were $k$ temporary shocks and the new shock is permanent.

Finally, considering the third case, if an outstanding shock reverses between $t$ and $t + dt$, by Bayes rule the firm's updated beliefs equal

$$p_k(t + dt) = \frac{p_{k+1}(t)(k+1)}{\sum_{i=1}^{n} p_i(t)\, i}. \tag{6}$$

When a shock reverses, the firm learns that it was temporary. Hence, there are $k$ outstanding temporary jumps after the reversal if and only if there were $k + 1$ outstanding temporary jumps before that. The joint probability of having $k + 1$ temporary jumps at time $t$ and observing a reversion between $t$ and $t + dt$ equals $p_{k+1}(k+1)\lambda_3 dt$. The probability of observing a reversion between $t$ and $t+dt$ conditional only on being at time $t$ equals $\sum_{i=1}^{n} p_i(t)\, i\lambda_3 dt$. Dividing the former probability by the latter yields (6).

Eqs. (4)-(6) fully characterize the dynamics of the firm's beliefs. Notice that reversion of a shock decreases the number of outstanding shocks, $n$, by one, while arrival of a new shock increases this number by one.

To keep the model both intuitive and tractable, in the early sections of the paper we focus on the case when there is only one shock, which can be either permanent or temporary.[10] In this case, the firm's beliefs are characterized by a single state variable

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[10] Section 5 considers the multi-shock setting.

$p(t) \equiv p_1(t)$, which is the probability that the outstanding shock is temporary. Before the arrival and after any reversion, there is no uncertainty about past shocks. At the moment $t_0$ of the arrival of the shock, the firm assesses the prior probability of the shock being temporary as $p(t_0) \equiv p_0 = \lambda_2 / (\lambda_1 + \lambda_2)$. After the shock arrives, the firm continuously updates its assessment of this probability according to (4):

$$\frac{dp(t)}{dt} = -\lambda_3 p(t)(1 - p(t)), \tag{7}$$

which is solved by

$$p(t) = \frac{\lambda_2}{\lambda_1 e^{\lambda_3 (t - t_0)} + \lambda_2}. \tag{8}$$

## 2.3 A Simple Case

Before presenting the details of our analysis in subsequent sections, we consider the simplest setting to illustrate the intuition behind our results. Suppose that there is only one possible shock and that the evolution of the cash flow does not have a Brownian motion component. In this case, the cash flow process is quite simple. At all times prior to the arrival of the shock, the cash flow stream is fixed at $X_0$. Upon the arrival of a shock it jumps to $X_0(1+\varphi)$, with $\varphi > 0$. With a permanent shock the cash flow remains at the higher level $X_0(1+\varphi)$ forever after, but with a temporary shock the cash flow eventually reverts to the level $X_0$ at some (random) point in the future.

Let us begin by calculating some simple values. If the firm knows for sure that the shock is permanent, then the value of investing is $\frac{X_0(1+\varphi)}{r} - I$, which is simply the present value of the perpetual cash flow $X_0(1+\varphi)$ minus the cost of investment. Conversely, if the firm knows for sure that the shock is temporary, then the value of investing is $\frac{X_0(1+\varphi+\lambda_3/r)}{r+\lambda_3} - I$, which is the net present value of receiving a flow of $X_0(1+\varphi)$ until the shock is reversed and $X_0$ thereafter.

To ensure that a non-trivial solution exists for the problem, we make the assumption that $X_0$ satisfies

$$\frac{rI}{1+\varphi} < X_0 < \frac{r + \lambda_3 p_0}{1 + \varphi + \frac{\lambda_3}{r} p_0} I. \tag{9}$$

Economically, the lower bound means that if the shock is known to be permanent, the net present value of investing into the project is positive, $\frac{X_0(1+\varphi)}{r} - I > 0$, or else there would never be any investment in this Bayesian setting. The upper bound, as we show later in this section, is equivalent to an assumption that learning has positive value. In particular, this assumption implies that if the shock is known to be temporary, then it is not optimal to invest: $\frac{X_0(1+\varphi+\lambda_3/r)}{r+\lambda_3} - I < 0$.

Let $G(p)$ denote the value of the option to invest, while a shock persists, and where $p$ is the current value of the belief process. We assume risk neutrality, where $r$ is the riskless rate of interest. Therefore, over the range of $p$ at which the option is not exercised, $G(p)$ must satisfy the equilibrium differential equation[11]:

$$(r + p\lambda_3) G(p) = -G'(p) \lambda_3 p(1-p) + p\lambda_3 H(X_0), \tag{10}$$

where $H(X_0)$ is the value of the option to invest after the shock reverts. By Assumption 1, the option will not be exercised after a temporary shock reverses, so $H(X_0) = 0$ and

$$(r + p\lambda_3) G(p) = -G'(p) \lambda_3 p(1-p). \tag{11}$$

The general solution to (11) is:

$$G(p) = C \cdot (1-p) \left(\frac{1}{p} - 1\right)^{\frac{r}{\lambda_3}}, \tag{12}$$

where $C$ is a constant to be determined by appropriate boundary conditions.

The option will be exercised when the conditional probability of the shock being temporary decreases to a lower threshold. Intuitively, it is optimal for the firm to invest only when it becomes sufficiently sure that the project will yield high cash flows for a long period of time. Let $\bar{p}$ denote the trigger at which the option is exercised. The exercise trigger $\bar{p}$ and constant $C$ are jointly determined by the following boundary conditions:

$$\begin{aligned} G(\bar{p}) &= (1-\bar{p}) \frac{X_0(1+\varphi)}{r} + \bar{p} \frac{X_0(1+\varphi+\lambda_3/r)}{r+\lambda_3} - I, & (13) \\ G_p(\bar{p}) &= -\frac{X_0(1+\varphi)}{r} + \frac{X_0(1+\varphi+\lambda_3/r)}{r+\lambda_3}. & (14) \end{aligned}$$

The first equation is the value-matching condition. It reflects the fact that upon exercise, the firm receives the conditional expected value of receiving $X_0(1+\varphi)$ forever if the shock is permanent, or of receiving $X_0(1+\varphi)$ until the shock is reversed and then $X_0$ thereafter if the shock is temporary. The second equation is the smooth-pasting condition.[12] It ensures that the trigger $\bar{p}$ maximizes the value of the investment option.

Combining (12) with (13) and (14) yields the optimal investment threshold $\bar{p}$ and the constant $C$:

$$\bar{p} = \frac{X_0(1+\varphi) - rI}{\lambda_3 \left(I - \frac{X_0}{r}\right)}, \tag{15}$$

$$C = \frac{1}{1-\bar{p}} \left(\frac{1}{\bar{p}} - 1\right)^{-\frac{r}{\lambda_3}} \left[(1-\bar{p}) \frac{X_0(1+\varphi)}{r} + \bar{p} \frac{X_0(1+\varphi+\lambda_3/r)}{r+\lambda_3} - I\right]. \tag{16}$$

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[11] See Dixit and Pindyck (1994), Chapter 3, for a derivation of the equilibrium differential equation for Poisson processes.

[12] This condition is also known as the high-contact condition (see Krylov (1980) and Dumas (1991) for a discussion).

Now we can see that the upper bound on $X_0$ from (9) is equivalent to assuming that $\bar{p} < p_0$. This restriction seems entirely reasonable since it ensures that there is some benefit to learning. Combined with the lower bound on $X_0$, this restriction guarantees that $\bar{p} \in (0, p_0)$. Also, note that in this simple case, the option will never be exercised prior to the arrival of the shock.[13] This is because the value of exercising prior to the arrival of the shock is dominated by the value of waiting until the shock arrives and then exercising immediately (which itself is dominated by waiting an additional period of time in order to learn).[14]

This simple case without the Brownian motion component highlights the key notion of Bayesian learning in a real options context. Perhaps the best way to gain intuition on the optimal investment policy is to re-write the expression for the optimal trigger $\bar{p}$ outlined in Eq. (15) as:

$$X_0 (1 + \varphi) = \bar{p}\lambda_3 \left( I - \frac{X_0}{r} \right) + rI. \tag{17}$$

The intuition behind expression (17) is the trade-off between investing now versus investing a moment later if the past shock persists, where the value of waiting is explicitly an "option to learn". If the firm invests now it gets the benefit of the cash flow $X_0 (1 + \varphi)$ over the next instant. This is the term on the left hand side of the equal sign. If the firm waits a moment and invests if the past shock persists, it faces a small chance of the shock reversing, in which case the expected value gained by not investing is equal to $\bar{p}\lambda_3 \left( I - \frac{X_0}{r} \right)$. Importantly, by waiting that additional moment, it gains the opportunity to forgo investment should the past shock prove to be temporary. The second term on the right hand side of the equal sign, $rI$, is the savings from delaying the investment cost by an instant. At the optimal Bayesian trigger, $\bar{p}$, these two sides are exactly equal, and the firm is indifferent between investing now and waiting a moment to learn.

Fig. 1 plots a simulated sample paths of the cash flow process, $X(t)$, and the firm's belief process, $p(t)$. Before the arrival of the shock, the investment is suboptimal since the project does not generate enough cash flows. When the shock arrives, the cash flow process jumps from $X$ to $X(1 + \varphi)$ and the net present value of the project becomes positive. Nevertheless, the firm finds investment suboptimal because of the valuable option to learn more about the nature of the past shock. As time goes by and the shock does not revert back, the firm updates its beliefs downwards. When the firm becomes sufficiently sure that the past shock is permanent, it invests. In the example in Fig. 1 this happens more than 1.5 years after the arrival of the shock.

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[13] This result will not hold in several generalized versions of the model that follow.

[14] To see this, note that the difference between exercising prior to the arrival of the shock and exercising upon the arrival of the shock is equal to the value of getting the cash flow $X_0$ until a shock occurs minus the cost savings between paying $I$ now and waiting until a shock arrives. This difference is equal to $\frac{X_0}{r+\lambda_1+\lambda_2} - \left(1 - \frac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2}\right) I$, or $\frac{X_0 - rI}{r+\lambda_1+\lambda_2}$, which is ensured to be negative by (9).

# 3 Model with Bayesian and Brownian Uncertainties

In the simple case in the previous section, the cash flow process $X(t)$ was a pure jump process. Since most traditional real options models are based on Brownian uncertainty, we generalize the simple case by allowing $X(t)$ to follow a combined Poisson and geometric Brownian motion process. In this case the firm faces two fundamentally different types of uncertainty. First, as in traditional models, the firm is uncertain about future shocks. This is captured by the cash flow process being subject to the Brownian innovations and unknown timing of the shock before the shock arrives. This gives rise to the option to wait for realizations of future shocks usually studied in the real options literature. Second, the firm is also uncertain about past shocks, meaning that the firm is unable to perfectly identify whether a past shock is permanent or temporary. This gives rise to the option to learn more about the nature of past shocks. In order to highlight the interaction of Bayesian and Brownian uncertainties, we assume only a single shock. We extend to the general case of multiple shocks in a subsequent section.

Since there is only one shock which can be either permanent or temporary, we solve the model by backward induction. First, we consider the optimal timing of investment conditional on a temporary shock having reversed. In this case, there is neither uncertainty about past shocks nor the possibility of new jumps, so the problem is the standard real options problem in the literature. Second, we move one step back and consider the situation after a shock arrives and the firm is uncertain about its nature. Finally, we consider the situation before a shock arrives.

## 3.1 Optimal Investment After a Shock Reverses

First, consider the situation after a temporary shock reverses. Then, there is no uncertainty about past shocks, so the only underlying state variable is the cash flow of the project, $X(t)$, where $X(t)$ follows a geometric Brownian motion:

$$dX(t) = \alpha X(t)dt + \sigma X(t)dB(t). \tag{18}$$

Let $H(X)$ denote the value of the investment option after the shock reverses, where $X$ is the current value of the cash flow process. Using standard arguments (e.g., Dixit and Pindyck (1994)), $H(X)$ must solve the following differential equation:

$$rH = \alpha X H_X + \frac{1}{2}\sigma^2 X^2 H_{XX},\ X \leq X^*. \tag{19}$$

Eq. (19) must be solved subject to the following boundary conditions:

$$H(X^*) = \frac{X^*}{r-\alpha} - I, \tag{20}$$

$$H'\left(X^*\right) = \frac{1}{r-\alpha}. \tag{21}$$

The first boundary condition is the value-matching which states that at the exercise time the value of the option equals to the net present value of the project. The second boundary condition is the smooth-pasting or high-contract condition that guarantees that the exercise strategy is chosen optimally. The last boundary condition is $H\left(0\right) = 0$ reflecting the fact that $X\left(t\right) = 0$ is the absorbing barrier for the cash flow process.[15]

Solving (19) subject to (20) and (21) yields the investment threshold $X^*$:

$$X^* = \frac{\beta}{\beta - 1} I,$$

where $\beta$ is the positive root of the fundamental quadratic equation $\frac{1}{2}\sigma^2\beta\left(\beta-1\right)+\alpha\beta-r = 0$:

$$\beta = \frac{1}{\sigma^2}\left[-\left(\alpha - \frac{\sigma^2}{2}\right) + \sqrt{\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}\right] > 1. \tag{22}$$

The corresponding value of the investment option $H\left(X\right)$ is then given by

$$H(X) = \begin{cases} \left(\frac{X}{X^*}\right)^\beta \left(\frac{X^*}{r-\alpha} - I\right) & if\ X < X^*, \\ \frac{X}{r-\alpha} - I & otherwise. \end{cases} \tag{23}$$

## 3.2 Optimal Investment While a Shock Persists

Consider the situation when there is an outstanding jump which can be either permanent or temporary. The process for $X(t)$ satisfies

$$dX\left(t\right) = \alpha X\left(t\right)dt + \sigma X\left(t\right)dB\left(t\right) - \mathbf{1}_{temp}\frac{\varphi X\left(t\right)}{1+\varphi}dN\left(t\right), \tag{24}$$

where $\mathbf{1}_{temp}$ is an indicator function of a temporary outstanding jump, $B(t)$ is a standard Wiener process, and $N(t)$ is a reversion process with intensity $\lambda_3$ which equals 1 after reversion occurs and 0 before.

We begin by calculating some simple values. If the firm knows for sure that a shock is permanent, then the value of investing immediately is $\frac{1}{r-\alpha}X - I$. If the firm knows for sure that the shock is temporary, then the value of investing immediately equals $\frac{1+\frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r-\alpha+\lambda_3}X - I$.

Let $G\left(X, p\right)$ denote the value of the option to invest while the shock persists, where $X$ and $p$ are the current values of the cash flow and the belief processes, respectively.

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[15] This boundary condition applies to all of the valuation equations. However, to avoid repetition, we do not list it in the future valuation equations.

Analogous to equation (10), in the range of $(X, p)$ at which the option is not exercised, $G(X, p)$ must satisfy the equilibrium partial differential equation:

$$(r + p\lambda_3)\, G = \alpha X G_X + \frac{\sigma^2}{2} X^2 G_{XX} - \lambda_3 p(1-p) G_p + p\lambda_3 H\left(\frac{X}{1+\varphi}\right), \tag{25}$$

where $H(X)$ is the value of the option when no more jumps can occur given by (23).

Exercise will be triggered when cash flows rise to an upper trigger, which is itself a function of the firm's beliefs. Let $\bar{X}(p)$ denote the exercise trigger function. We show in the appendix that $\bar{X}(p) < X^*(1+\varphi)$. Equation (25) is solved subject to the following value-matching and smooth-pasting conditions:

$$G\left(\bar{X}(p), p\right) = \left[(1-p)\frac{1}{r-\alpha} + p\frac{1 + \frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r - \alpha + \lambda_3}\right]\bar{X}(p) - I, \tag{26}$$

$$G_X(\bar{X}(p), p) = \left[(1-p)\frac{1}{r-\alpha} + p\frac{1 + \frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r - \alpha + \lambda_3}\right], \tag{27}$$

$$G_p(\bar{X}(p), p) = \left[-\frac{1}{r-\alpha} + \frac{1 + \frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r - \alpha + \lambda_3}\right]\bar{X}(p). \tag{28}$$

Intuitively, the value-matching condition (26) captures the fact that at the time of investment the value of the investment option equals the expected payoff from immediate investment, while the smooth-pasting conditions (27) - (28) guarantee that the trigger function is chosen optimally.

Evaluating (25) at $\bar{X}(p)$, plugging in the boundary conditions (26) - (28) and simplifying, provides the following expression for the optimal trigger function $\bar{X}(p)$:[16]

$$\begin{aligned}\bar{X}(p) = p\lambda_3 &\left[\left(\tfrac{\bar{X}(p)}{(1+\varphi)X^*}\right)^{\beta}\left(\tfrac{X^*}{r-\alpha} - I\right) - \left(\tfrac{\bar{X}(p)}{(1+\varphi)(r-\alpha)} - I\right)\right] \\ &+ rI + \tfrac{\sigma^2}{2}\bar{X}(p)^2 G_{XX}\left(\bar{X}(p), p\right).\end{aligned} \tag{29}$$

In comparing the expression for $\bar{X}(p)$ to the trigger function without Brownian uncertainty in Eq. (17), we find that we now have the same general form of the trigger, plus a new convexity term, $\frac{\sigma^2}{2}\bar{X}(p)^2 G_{XX}\left(\bar{X}(p), p\right)$. Here, the trigger equals the sum of the value of the option to learn to see if the shock reverts and a convexity term that represents the traditional option to wait in the real options literature.[17] In this sense,

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[16] In general, $\bar{X}(p)$ and $G(X, p)$ can be computed numerically. In the appendix we also derive the closed-form solution for the special case of $\sigma = 0$.

[17] In order to ensure optimality of the exercise trigger, $\bar{X}(p)$, $G(X, p)$ must be convext at $\bar{X}(p)$. To see this, note that for a given $p$, $G(X, p) < h(p)X - I$ for all $X < \bar{X}(p)$, where $h(p) = \left[(1-p)\frac{1}{r-\alpha} + p\frac{1+\frac{\lambda_3}{(r-\alpha)(1+\phi)}}{r-\alpha+\lambda_3}\right]$. From the value-matching condition, at $\bar{X}(p)$, $G(\bar{X}(p), p) = h(p)\bar{X}(p) - I$, and from the smooth-pasting condition $G_X(\bar{X}(p), p) = h(p)$. Thus, at $\bar{X}(p)$, it must be the case that $G_{XX}(\bar{X}(p), p) > 0$.

Brownian uncertainty is additive to Bayesian uncertainty. In other words, the addition of Brownian uncertainty to the model increases the investment trigger by a further component due to the option value of waiting for the evolution of Brownian uncertainty over future cash flows.

It is clear that the trigger function $\bar{X}(p)$ is increasing in $p$. For any two values of $p$ at exercise, the expected payoff from exercise is higher for the case of the lower value of $p$. Thus for lower values of $p$, exercise will occur earlier due to the higher expected payoff. Given this monotonicity, we can invert the function and also express the exercise trigger by the function $\bar{p}(X)$. In this formulation, the firm's investment strategy can be characterized in the following way. At any time $t$, given the current value of the cash flow process $X(t)$, the firm compares its beliefs $p(t)$ with the boundary level $\bar{p}(X(t))$ and invests at the first instant when $p(t)$ becomes lower than $\bar{p}(X(t))$.

The quantitative effects of the addition of Brownian uncertainty to the model are illustrated by Fig. 2 which shows the investment trigger function $\bar{X}(p)$ for different values of the volatility parameter.[18] Both Bayesian and Brownian uncertainties lead to a significant increase in the investment trigger. If there were no Brownian or Bayesian uncertainty and all shocks were permanent, the trigger would equal 1. Now, consider the case in which we add only Bayesian uncertainty, corresponding to the bottom curve in which $\sigma = 0$. Consider the case of $p = 2/3$, equaling the value of $p_0$ for our parameter specification. Here we find that the impact of pure Bayesian uncertainty increases the investment threshold by 12.6%, which is labeled as the "Bayesian effect" in Fig. 2. Now, consider the additional impact of Brownian uncertainty. The middle and top curves correspond to the cases of $\sigma = 0.05$ and $\sigma = 0.10$, respectively. The addition of Brownian uncertainty leads to an *additional* increase in the threshold by 5.6% for the case of $\sigma = .05$ and by 20% for $\sigma = .10$, which are labeled as the "Brownian effects" in Fig. 2.[19] As discussed above, the addition of Brownian uncertainty does not undo the effect that Bayesian learning has on the optimal exercise rule. Indeed, the shape of the trigger function $\bar{X}(p)$ does not change much as the Brownian volatility parameter $\sigma$ increases. For any $\sigma$, $\bar{X}(p)$ is an increasing and concave function of $p$, and the value of the option to learn is significant.

## 3.3 Optimal Investment Prior to the Arrival of a Shock

To complete the solution, consider the value of the investment option before the jump occurs. In this case, the only underlying uncertainty concerns the future path of $X(t)$.

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[18] To compute the trigger functions we used a variation of the least-squares method developed by Longstaff and Schwartz (2003). The procedure is outlined in the appendix.

[19] Note that this implies that the total instantaneous volatility of the cash flow process is higher than $\sigma$, since it also includes volatility due to the jumps.

Specifically, $X(t)$ evolves according to

$$dX(t) = \alpha X(t)\,dt + \sigma X(t)\,dB(t) + \varphi X(t)\,dM(t)\,, \tag{30}$$

where $M(t)$ is an arrival process with intensity $\lambda_1 + \lambda_2$ and which equals 1 after a shock occurs and 0 before that.

Denote the option value by $F(X)$, where $X$ is the current value of the state variable. Prior to the investment, $F(X)$ solves

$$(r + \lambda_1 + \lambda_2) F(X) = \alpha X F'(X) + \frac{1}{2}\sigma^2 X^2 F''(X) + (\lambda_1 + \lambda_2) G(X(1+\varphi), p_0)\,, \tag{31}$$

where $G(X, p)$ is the value of the investment option when the shock persists. If the investment option is exercised prior to the arrival of a shock, the firm receives:

$$\frac{X}{r-\alpha} + \frac{\varphi X}{r - \alpha + \lambda_1 + \lambda_2}\left(\frac{\lambda_1}{r-\alpha} + \frac{\lambda_2}{r - \alpha + \lambda_3}\right) - I. \tag{32}$$

The first term of (32) is the discounted cash flows that the firm gets if the jump never occurs, while the last term is the investment cost that the firm needs to incur to launch the project. The two other terms correspond to the additional cash flows the firm gets from the shocks. If the shock is permanent, the firm gets additional expected cash flows of $\frac{\varphi X e^{\alpha\tau}}{(r-\alpha)}$ at the time of the shock $\tau$. If the shock turns out to be temporary, the additional expected cash flows at the time of the shock $\tau$ are $\frac{\varphi X e^{\alpha\tau}}{(r-\alpha+\lambda_3)}$. Integrating over $\tau$ yields the second and third terms of (32).

Let $\hat{X}$ denote the optimal investment trigger before the arrival of the shock. Conjecture that it is strictly optimal to wait when $X(t)$ is below $\bar{X}(p_0)/(1+\varphi)$, that is, $\hat{X} \geq \bar{X}(p_0)/(1+\varphi)$. This is quite intuitive, in that it implies that if it isn't optimal to invest at an instant prior to the shock, then it would not be optimal to invest if the jump occurs and immediately reverses itself. This is confirmed in the appendix. Then, the investment option value $F(X)$ can be divided into two parts $F_L(X)$ and $F_H(X)$, corresponding to the lower and the higher regions, respectively. By Itô's lemma, $F_L(X)$ and $F_H(X)$ satisfy the following differential equations:

- in the region $X < \bar{X}(p_0)/(1+\varphi)$,

$$(r + \lambda_1 + \lambda_2) F_L(X) = \alpha X F_L'(X) + \frac{1}{2}\sigma^2 X^2 F_L''(X) + (\lambda_1 + \lambda_2) G(X(1+\varphi), p_0)\,; \tag{33}$$

- in the region $\bar{X}(p_0)/(1+\varphi) < X < \hat{X}$,

$$\begin{aligned}(r + \lambda_1 + \lambda_2) F_H(X) &= \alpha X F_H'(X) + \tfrac{1}{2}\sigma^2 X^2 F_H''(X) \\ &+ \left(\lambda_1 \tfrac{1+\varphi}{r-\alpha} + \lambda_2 \tfrac{1+\varphi+\frac{\lambda_3}{r-\alpha}}{r-\alpha+\lambda_3}\right) X - (\lambda_1 + \lambda_2) I.\end{aligned} \tag{34}$$

The differential equations (33) and (34) differ due to the implied investment behavior at the moment of the arrival of a shock. In the lower region the arrival of a shock does not induce immediate investment, while in the higher region the arrival of a shock implies immediate investment.

Differential equations (33) and (34) are solved subject to the following boundary conditions:

$$F_H\left(\hat{X}\right) = \frac{\hat{X}}{r-\alpha} + \frac{\varphi \hat{X}}{r-\alpha+\lambda_1+\lambda_2}\left(\frac{\lambda_1}{r-\alpha} + \frac{\lambda_2}{r-\alpha+\lambda_3}\right) - I, \tag{35}$$

$$F_H'\left(\hat{X}\right) = \frac{1}{r-\alpha} + \frac{\varphi}{r-\alpha+\lambda_1+\lambda_2}\left(\frac{\lambda_1}{r-\alpha} + \frac{\lambda_2}{r-\alpha+\lambda_3}\right), \tag{36}$$

$$\lim_{X\uparrow \bar{X}(p_0)/(1+\varphi)} F_L(X) = \lim_{X\downarrow \bar{X}(p_0)/(1+\varphi)} F_H(X), \tag{37}$$

$$\lim_{X\uparrow \bar{X}(p_0)/(1+\varphi)} F_L'(X) = \lim_{X\downarrow \bar{X}(p_0)/(1+\varphi)} F_H'(X), \tag{38}$$

$$\lim_{X\to 0} F_L(X) = 0. \tag{39}$$

As before, the value-matching condition (35) imposes equality at the exercise point between the value of the option and the expected payoff from immediate investment, while the smooth-pasting condition (36) ensures that the exercise point is chosen optimally. Conditions (37) and (38) guarantee that the value of the investment option is continuous and smooth. Finally, (39) is a boundary condition that reflects the fact that $X = 0$ is the absorbing barrier for the cash flow process.

In the appendix we show that combining the system of differential equations (33)-(34) with the boundary conditions (35)-(39) yields the following investment trigger $\hat{X}$:

$$\begin{aligned}\hat{X} = {} & \tfrac{\gamma_1}{\gamma_1-1}\tfrac{(r-\alpha+\lambda_1+\lambda_2)}{r+\lambda_1+\lambda_2}\left[rI + \left(\tfrac{\bar{X}(p_0)}{(1+\varphi)\hat{X}}\right)^{-\gamma_2}(\lambda_1+\lambda_2)\, I\right] \\ & + \left(\tfrac{\bar{X}(p_0)}{(1+\varphi)\hat{X}}\right)^{-\gamma_2}\left[-\left(\tfrac{\lambda_1}{r-\alpha} + \tfrac{\lambda_2+\frac{\lambda_2\lambda_3}{(1+\varphi)(r-\alpha)}}{r-\alpha+\lambda_3}\right)\bar{X}(p_0) + \tfrac{2(\lambda_1+\lambda_2)(r-\alpha+\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-1)}\Gamma\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)\right],\end{aligned} \tag{40}$$

where $\gamma_1$ and $\gamma_2$ are the positive and negative roots the fundamental quadratic equation $\frac{1}{2}\sigma^2\gamma(\gamma-1) + \alpha\gamma - r - \lambda_1 - \lambda_2 = 0$, and

$$\Gamma(X) \equiv X^{\gamma_2}\int \frac{G(X(1+\varphi), p_0)}{X^{\gamma_2+1}}dX. \tag{41}$$

The corresponding value of the investment option $F(X)$ is shown in the appendix.

We can now fully summarize the optimal investment strategy in the following proposition:

**Proposition 1.** *The optimal investment strategy for the model with both Bayesian and Brownian uncertainty is:*

1. *If the shock does not occur until* $X(t)$ *reaches* $\hat{X}$*, then it is optimal for the firm to invest when* $X(t) = \hat{X}$*;*

2. *If the shock occurs before the point when* $X(t)$ *reaches* $\hat{X}$*, and at the time of the shock* $X(t)$ *is above* $\bar{X}(p_0) / (1+\varphi)$*, then it is optimal for the firm to invest immediately after the shock;*

3. *If the shock occurs before* $X(t)$ *reaches* $\hat{X}$*, and at the time of the shock* $X(t)$ *is below* $\bar{X}(p_0) / (1+\varphi)$*, then it is optimal for the firm to invest at the first time when* $X(t) = \bar{X}(p(t))$*;*

4. *If the shock occurs before* $X(t)$ *reaches* $\hat{X}$*, at the time of the shock* $X(t)$ *is below* $\bar{X}(p_0) / (1+\varphi)$*, and it reverts back before* $X(t)$ *reaches* $\bar{X}(p(t))$*, then it is optimal to invest when* $X(t) = X^*$*.*

## 3.4 Discussion

Proposition 1 demonstrates that there are four fundamentally different scenarios for the firm's investment timing. Under the first scenario, the cash flow process $X(t)$ increases up to $\hat{X}$ before the jump occurs. Prior to the arrival of the shock there is no learning, so the investment trigger is constant over time. One simulated sample path that satisfies this scenario is shown in the upper left corner of Fig. 3.

Under the other three possible scenarios, the shock occurs before the cash flow process reaches $\hat{X}$, so the firm does not invest prior to the arrival of the shock. After the arrival of the shock, the investment trigger is a function of the firm's beliefs about the past shock given by (29). Thus, as time passes and the shock persists, the firm learns more about the nature of the past shock. When the firm observes that the shock does not revert back, it updates (lowers) its assessment of the probability that the past shock was temporary. Since immediate investment is more attractive when the past shock is permanent, the investment trigger $\bar{X}(p(t))$ decreases over time.

Under the second scenario, the value of the cash flow process immediately after the shock overshoots $\bar{X}(p_0)$. In this case, the investment occurs immediately after the shock arrives. A simulated sample path satisfying this scenario is shown in the upper right corner of Fig. 3.

In the third scenario, the cash flow process $X(t)$ reaches the investment trigger $\bar{X}(p(t))$ following a period of learning, but prior to any potential reversion of the shock. A simulated sample path illustrating this scenario is shown in the lower left corner of Fig. 3. Notice that this graph provides an illustration of several interesting properties of investment in a Bayesian setting such as the dependence on the timing of past shocks and the sluggish response of investment to shocks. We discuss these and other implications in more details in the next section.

Finally, under the fourth scenario the past temporary shock reverts back before the firm invests. If this happens, the problem becomes standard. After the reversal, the investment trigger is constant over time at the level $X^*$. The firm invests at the first time when the cash flow process $X(t)$ reaches $X^*$. A simulated sample path that describes the fourth scenario is shown in the lower right corner of Fig. 3.

# 4 Model Implications

In this section we analyze some important implications of the model.

## 4.1 Bayesian Uncertainty and the Valuable Option to Learn

In traditional real options models focusing on uncertainty about future shocks only, investment takes place when cash flows rise to a constant trigger. This is due to the fact that uncertainty is entirely due to the Brownian component which is constant over time. This trigger embodies the notion of the traditional option to wait, as investment is triggered when the real option is sufficiently in the money.

As we add the Bayesian feature of uncertainty over past shocks, the trigger, $\bar{X}(p(t))$, is no longer constant, but is a function of the firm's beliefs about the nature of past shocks. More specifically, the investment trigger is high soon after the arrival of a shock, but then goes down as time goes by. Thus, in the Bayesian framework, uncertainty is no longer constant, but declines over time as the firm becomes progressively more confident in a past shock's permanence. Thus, in addition to the standard option to wait, we now also have a valuable option to learn.

We see in Eq. (29) that the investment trigger exceeds the Marshallian trigger of $rI$ due to two forces: the option to learn (and waiting until a moment later and seeing if a past shock reverses) and the option to wait (and benefit from cash flows rising over the next moment). As demonstrated in Fig. 2, these forces are additive, and appear to be of similar magnitudes. Ignoring either option would appear to lead to a similar investment timing error.

The combination of the Bayesian and Brownian options leads to further investment delay, and thus greater sluggishness in responding to positive cash flow shocks. For example, given the additional Bayesian uncertainty over past shocks, a firm is now less likely to invest soon after the arrival of a positive shock, where the option to learn is its most valuable. Then, as time goes by, uncertainty is reduced, and the firm is willing to invest at a lower level of the cash flow process.[20]

The slow response of investment to cash flow shocks generated by the option to learn is consistent with the slow response of economic variables such as investment, labor demand

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[20] See Bloom (2008) for the evidence that such uncertainty shocks as the 1973 oil price shock have large aggregate effects due to more "cautious" behavior by economic agents.

and prices to shocks (e.g., Caballero and Engel (2004)). Another stylized fact which is consistent with Bayesian learning about past shocks is that the response of investment to shocks is time-dependent. In particular, aggregate investment is shown to be more responsive to shocks if the period preceding expansion is longer (Caballero et al. (1995), Bachmann et al. (2008)). Our argument appears consistent with this evidence since firms are likely to believe that positive shocks following the period of preceding expansion are more likely to be permanent, so the option to learn is less valuable in this case. The result that uncertainty about the identity of the shock can lead to sluggish response of investment to shocks is very general and can hold in different environments with shocks of different types. For example, Moore and Schaller (2002) show in simulations that uncertainty about the persistence of interest rate shocks can lead to the sluggishness of investment in the context of the neoclassical q-theory.

Note that the magnitude of the slow response of investment to cash flow shocks depends on two key parameters: the size of the shock $\varphi$ and the speed of learning $\lambda_3$. The papers that calibrate models of investment decisions usually assume that temporary shocks are relatively persistent. For example, Thomas (2002) and Gourio and Kashyap (2007) assume that a productivity shock follows an AR(1) process with coefficient 0.9225. In the context of our model this would mean that learning about the identity of a shock would be rather slow. This suggests that incorporating uncertainty about whether shocks are permanent or temporary into the models of aggregate investment can have important quantitative effects.

## 4.2 Investment in the Face of Stable or Declining Cash Flows

Another interesting property of our Bayesian model is that investment may occur in the face of stable or even decreasing cash flows. Since standard real options models imply that investment will be triggered when shocks push the underlying cash flow level up to a fixed upper threshold, the firm can rationally exercise the investment option only when the underlying cash flows increase. However, when the traditional real options model is extended to include the Bayesian feature, the firm may invest even when the cash flow process is stable or declining simply because it has become more certain about the permanent nature of the past shocks.

As an illustration of this property, consider a particular sample path of cash flows and the investment response presented in Fig. 4. We see that investment is made at a time of steady cash flows. At the moment of investment, it was the reduction in Bayesian uncertainty rather than the increase in current cash flows that triggered investment. This is made possible by the monotonically decreasing trigger function, $\bar{X}\left(p\left(t\right)\right)$.

Empirically, we certainly see examples of firms (and industries) investing in markets where cash flows are stable, or even declining. As detailed in Grenadier (1996), during the late 1970's and early 1980's several U.S. cities saw explosive growth in office building development in the face of rapidly increasing office vacancy rates. Specifically, consider the cases of the Denver and Houston office markets. Over the thirty-year period from 1960 through 1990, over half of all office construction was completed in a four-year interval: 1982-1985. Since office space takes an estimated average length of time between the initiation and completion of construction of 2.5 years, this investment was likely initiated over the period from 1979-1983. Throughout this period, office vacancies in these two cities were around 30%, considerably above previous levels. Notably, these two cities (as well as most of the cities experiencing unprecedented office growth during this period) were oil-patch cities where developers likely concluded (incorrectly) that high oil prices in the late 1970's and early 1980's would last indefinitely, as discussed in the quote provided in the Introduction. In other words, our model explains this period of rapid construction in times of declining cash flows by saying that the firms became very sure that high oil prices of 70s and early 80s, and hence, high rents in these oil cities will last forever.

Fig. 4 also makes evident that accounting for uncertainty about past shocks may lead to a failure of the "record-setting news principle". The record-setting news principle is that investment occurs only at instants in which the value of the cash flow process is the highest in its whole history. In Fig. 4 we see that investment occurs at a cash flow level that is more than 5% lower than its previous maximum. While the record-setting news principle holds for a large class of real options models (Boyarchenko (2004)), the addition of Bayesian uncertainty over the nature of past shocks can overturn this result. Thus, even when current cash flows are not at their all-time maximum, the decline in uncertainty about past shocks can trigger investment.

## 4.3 Investment Timing and the Maturity Structure of Project Cash Flows

In the standard real options literature, there is a simple equivalence between options that pay off in cash flows and those that pay off with an identical lump sum value. For example, Chapter 5 in Dixit and Pindyck (1994) considers the optimal exercise rule for options that pay off with a lump sum value of $V(t)$. Then, in Chapter 6, they perform an analogous analysis for options that pay off with a perpetuity cash flow of $P(t)$, with identical present value to the lump sum value $V(t)$. They show that the optimal exercise rules are identical.

However, in the context of valuations that are driven by the possibility of both temporary and permanent shocks, the timing of cash flows can be quite important. The greater the "front-loadedness" of the option payoff, the less important is the assessment of the relative likelihood that a shock is temporary or permanent. In this section we consider a simple parameterization of the front-loadedness of the option payoff, ranging from payoffs that are equivalent to a one-time lump sum to payoffs that are equivalent to perpetual cash flows.

Consider the simplest case studied in Section 2.3, with one alteration. Assume now that if an option is exercised at time $\tau$, it provides a stream of payments $\left(1+\frac{k}{r}\right)e^{-k(t-\tau)}X(t)$, $t \geq \tau$. Parameter $k \in [0,+\infty)$ captures the degree of front-loadedness of the project cash flows. Projects with low values of $k$ are relatively back-loaded: much of their cash flows are generated long after the exercise time. High values of $k$ mean that the project is relatively front-loaded, with most cash flows coming relatively close to the exercise time. The particular parameterization was chosen so as to make the present value of cash flows from the immediate exercise of the project in the no-shock case independent of $k$: $\displaystyle\int_{\tau}^{\infty} X(\tau)\left(1+\frac{k}{r}\right)e^{-k(t-\tau)}e^{-r(t-\tau)}dt = \frac{X(\tau)}{r}$. Of course, other reasonable parameterizations are possible.

This specification of cash flows captures two cases widely used in the real options literature. First, when $k=0$, the model reduces to the one studied in Section 2.3. In this case, the project pays a perpetual flow of $X(t)$ upon exercise. Second, if $k \to \infty$, payments from the project converge to a one time lumpy payment of $\frac{X(\tau)}{r}$ at the time of exercise $\tau$.

Similar to (9), to ensure that the project has a potentially positive net present value and that there is positive value to learning we make the assumption that $X_0$ satisfies

$$\frac{rI}{1+\varphi} < X_0 < \frac{r+\lambda_3 p_0}{1+\varphi+\frac{\lambda_3}{r}p_0+\frac{\lambda_3}{r}p_0\frac{k\varphi}{(r+\lambda_3+k)}}I. \tag{42}$$

Compared to (9), (42) puts the same lower bound and a more restrictive upper bound. As previously, these bounds guarantee that the solution to the investment timing problem is non-trivial.

As in Section 2.3, let the value of the option while the shock persists be denoted by $G(p)$. Over the range of $p$ at which the option is not exercised, the standard argument implies

$$(r+p\lambda_3)\,G(p) = -G'(p)\lambda_3 p(1-p). \tag{43}$$

This equation has the general solution

$$G(p) = C_1\left(1-p\left(t\right)\right)\left(\frac{1}{p}-1\right)^{\frac{r}{\lambda_3}}, \tag{44}$$

where $C_1$ is some constant. Note this general solution coincides with the one obtained in Section 2.3, equation (12). However, because of the more general cash flow timing assumption, the boundary conditions are now different.

Because the payoff from the project if the current shock is temporary is affected by the parameter $k$, the value-matching condition at the exercise trigger $\bar{p}_k$ is now:

$$G\left(\bar{p}_k\right) = \left(1-\bar{p}_k\right)\frac{X\left(1+\varphi\right)}{r} + \bar{p}_k\frac{X\left(1+\varphi+\frac{\lambda_3}{r}+\frac{k(1+\varphi)}{r}\right)}{r+\lambda_3+k} - I. \tag{45}$$

Note that although the present value of the firm's cash flow in case of the permanent jump does not depend on $k$, in case of the temporary jump it depends on $k$ positively. Intuitively, a more front-loaded project allows the firm to capture more of the temporary high cash flows than a more back-loaded project.

As in Section 2.3, the exercise trigger is chosen to maximize the value of the option (or equivalently, to satisfy the smooth-pasting condition), giving the resulting optimal trigger value:

$$\bar{p}_k = \frac{X(1+\varphi) - rI}{\lambda_3 \left( I - \frac{X}{r} - \frac{Xk\varphi}{r(r+\lambda_3+k)} \right)}. \tag{46}$$

Note, when $k = 0$, $p_k = \bar{p}_0$ gives us the trigger we obtained in Section 2.3.

Again, given that there is value to learning, it is straightforward to show that the option will never be exercised prior to the arrival of the shock. Thus, the optimal investment rule is indeed for the firm to invest at the first moment that the posterior probability $p(t)$ falls to the trigger $\bar{p}_k$, and never if the trigger is not reached.

Consider how the parameter of front-loadedness affects the trigger value:

$$\frac{\partial \bar{p}_k}{\partial k} = \frac{r\varphi(r+\lambda_3)}{\lambda_3} \frac{X(1+\varphi) - Ir}{\left[X\left(r+\lambda_3+k(1+\varphi)\right) - Ir\left(r+\lambda_3+k\right)\right]^2} X > 0. \tag{47}$$

We therefore find that the greater the front-loadedness, the earlier the option is exercised. In contrast, learning has greater value for projects whose payoffs arrive further in the future.

Intuitively, if the project cash flows are very front-loaded, then knowing the identity of the shock is not very important for the firm. In this case the firm gets a large part of the project cash flows very soon after the exercise date. Because of that, the learning option is not very valuable, so the firm invests early. On the other hand, if the project cash flows are very back-loaded, then it is very important for the firm to be sure that past shocks are permanent. As a result, the learning option is very valuable and the firm invests much later.

# 5 A Model with an Unlimited Number of Shocks

In this section, we analyze our most general version of the model: one with an unlimited number of shocks, as well as both Brownian and Bayesian uncertainty.[21]

Specifically, suppose that at any point in time the firm can have any number $n = 0, 1, \ldots$ of shocks outstanding. As before, the reversal of each outstanding temporary shock occurs with intensity $\lambda_3$ independently of all other processes in the economy. In addition, at each

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[21] The authors have also solved the model for any finite number of potential shocks. The results are very similar to those presented for the case of a countably infinite number of shocks, but with some additional notational burdens.

time a new permanent shock occurs with intensity $\lambda_1$, and a new temporary shock occurs with intensity $\lambda_2$.

Given our assumptions, at each time $t$ the state of the economy can be described by a pair of variables $(X(t), p(t))$, where $X(t)$ is the current value of the cash flow process and $p(t) = (p_1(t), p_2(t), \ ...)'$ is the infinitely dimensional vector of the firm's beliefs. Specifically, $p_k(t)$ is the probability at time $t$ that there are $k$ outstanding temporary jumps. Obviously, the firm's belief at time $t$ that there are no outstanding temporary jumps equals $1 - \iota' p(t)$, where $\iota$ is the infinitely dimensional vector of ones. The initial state can be described by $(X(0), p(0)) = (X(0), \mathbf{0})$, and at any time there exists $\hat{k}$ such that for all $k > \hat{k}$, $p_k(t) = 0$. Intuitively, $\hat{k}$ corresponds to the total number of outstanding shocks.

As in Section 2.1, the evolution of the cash flow process is given by

$$dX(t) = \alpha X(t)\,dt + \sigma X(t)\,dB(t) + \varphi X(t)\,dM(t) - \frac{\varphi X(t)}{1+\varphi} dN_k(t), \ X(0) = X_0 \tag{48}$$

where $dB(t)$ is the increment of a standard Wiener process, and $dM(t)$ and $dN_k(t)$ are independent Poisson processes corresponding to the arrival and reversal of jumps, respectively. The intensity of $dM(t)$ is $\lambda_1 + \lambda_2$ and the intensity of $dN_k(t)$ is $k\lambda_3$, where $k$ is the number of outstanding temporary shocks.

From Section 2.1, and letting the number of potential shocks $n$ go to infinity, we have the following evolution of the Bayesian learning process. If no new shocks occur over the period from $t$ to $t + dt$:

$$\frac{dp_k(t)}{dt} = -\lambda_3 p_k(t)\left(k - \sum_{i=1}^{\infty} p_i(t)\, i\right). \tag{49}$$

If a new shock occurs between $t$ and $t + dt$, then the updated beliefs equal

$$p_k(t+dt) \equiv \hat{p}_k(p(t)) = \left\{ \begin{array}{ll} p_{k-1}(t)\, p_0 + p_k(t)(1-p_0) & \text{for } k = 2, 3, ..., \\ (1 - \iota' p(t))\, p_0 + p_1(t)(1-p_0) & \text{for } k = 1. \end{array} \right. \tag{50}$$

If an outstanding shock reverses between $t$ and $t + dt$, the updated beliefs equal

$$p_k(t+dt) \equiv \tilde{p}_k(p(t)) = \frac{p_{k+1}(t)(k+1)}{\sum_{i=1}^{\infty} p_i(t)\, i} \text{ for } k = 1, 2, .... \tag{51}$$

Let $S(X, p)$ denote the value of the underlying project. It is the expected discounted value of cash flows that the firm gets if it immediately exercises the investment option. Using the standard arguments, $S(X, p)$ must satisfy:

$$\begin{aligned} \left(r + \lambda_1 + \lambda_2 + \lambda_3 \textstyle\sum_{i=1}^{\infty} p_i i\right) S(X, p) &= \alpha X S_X + \tfrac{\sigma^2}{2} X^2 S_{XX} - \lambda_3 \textstyle\sum_{i=1}^{\infty} \tfrac{\partial S}{\partial p_i} p_i \left(i - \textstyle\sum_{j=1}^{\infty} p_j j\right) \\ &+ (\lambda_1 + \lambda_2)\, S\left(X(1+\varphi), \hat{p}(p)\right) + \left(\lambda_3 \textstyle\sum_{i=1}^{\infty} p_i i\right) S\left(\tfrac{X}{1+\varphi}, \tilde{p}(p)\right) + X. \end{aligned} \tag{52}$$

The solution can be written as

$$S\left(X,p\right)=a_0X+\sum_{i=1}^{\infty}p_i\left(a_i-a_0\right)X, \tag{53}$$

where constants $a_0, a_1, ...$ are defined in Eq. (75) in the appendix.

Let $G\left(X,p\right)$ denote the value of the investment option. Again, using standard arguments, prior to exercise $G\left(X,p\right)$ must satisfy:

$$\begin{aligned}\left(r+\lambda_1+\lambda_2+\lambda_3\textstyle\sum_{i=1}^{\infty}p_i i\right)G\left(X,p\right)=\alpha XG_X+\tfrac{\sigma^2}{2}X^2G_{XX}-\lambda_3\textstyle\sum_{i=1}^{\infty}\frac{\partial G}{\partial p_i}p_i\left(i-\sum_{j=1}^{\infty}p_j j\right)\\+\left(\lambda_1+\lambda_2\right)G\left(X\left(1+\varphi\right),\hat{p}\left(p\right)\right)+\left(\lambda_3\textstyle\sum_{i=1}^{\infty}p_i i\right)G\left(\tfrac{X}{1+\varphi},\tilde{p}\left(p\right)\right).\end{aligned} \tag{54}$$

The optimal investment decision can be described by a trigger function $\bar{X}\left(p\right)$. Eq. (54) is solved subject to the following value-matching and smooth-pasting conditions:

$$\begin{gathered}G\left(\bar{X}\left(p\right),p\right)=S\left(\bar{X}\left(p\right),p\right)-I,\\G_X\left(\bar{X}\left(p\right),p\right)=S_X\left(\bar{X}\left(p\right),p\right),\\\left(\lambda_3\textstyle\sum_{i=1}^{\infty}\left(\frac{\partial G\left(\bar{X}\left(p\right),p\right)}{\partial p_i}-\frac{\partial S\left(\bar{X}\left(p\right),p\right)}{\partial p_i}\right)\right)p_i\left(i-\sum_{j=1}^{\infty}p_j j\right)=0.\end{gathered} \tag{55}$$

The first equation is the value-matching condition. The second and third equations are the smooth-pasting conditions (with respect to $X$ and $t$, respectively).

Combining (52), (54) and (55) gives us[22]:

$$\begin{aligned}\bar{X}\left(p\right)-rI=\left(\lambda_1+\lambda_2\right)\left[G\left(\bar{X}\left(p\right)\left(1+\varphi\right),\hat{p}\left(p\right)\right)+I-S\left(\bar{X}\left(p\right)\left(1+\varphi\right),\hat{p}\left(p\right)\right)\right]\\+\left(\lambda_3\textstyle\sum_{i=1}^{\infty}p_i i\right)\left[G\left(\tfrac{\bar{X}\left(p\right)}{1+\varphi},\tilde{p}\left(p\right)\right)+I-S\left(\tfrac{\bar{X}\left(p\right)}{1+\varphi},\tilde{p}\left(p\right)\right)\right]\\+\tfrac{\sigma^2}{2}\bar{X}\left(p\right)^2G_{XX}\left(\bar{X}\left(p\right),p\right).\end{aligned} \tag{56}$$

When the state of the economy $\left(X,p\right)$ is very close to the trigger $\bar{X}\left(p\right)$, it is clear that the arrival of new positive shock will result in immediate investment. The intuition is simple. If the firm is indifferent between investing and waiting today, an upward jump in $X$ will certainly induce the firm to invest immediately. This implies:

$$G\left(\bar{X}\left(p\right)\left(1+\varphi\right),\hat{p}\left(p\right)\right)=S\left(\bar{X}\left(p\right)\left(1+\varphi\right),\hat{p}\left(p\right)\right)-I. \tag{57}$$

Therefore,

$$\begin{aligned}\bar{X}\left(p\right)&=\left(\lambda_3\sum_{i=1}^{\infty}p_i i\right)\left[G\left(\frac{\bar{X}\left(p\right)}{1+\varphi},\tilde{p}\left(p\right)\right)+I-S\left(\frac{\bar{X}\left(p\right)}{1+\varphi},\tilde{p}\left(p\right)\right)\right]+rI\\&\quad+\frac{\sigma^2}{2}\bar{X}\left(p\right)^2G_{XX}\left(\bar{X}\left(p\right),p\right).\end{aligned} \tag{58}$$

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[22] Note that since $S(X,p)$ is linear in $X$, $S_{XX}=0$.

Given the similarity between (58) and (29), it becomes clear how the multi-shock case generalizes the single-shock case. The intuition behind expression (58) is the trade-off between investing now versus investing a moment later if all of the past shocks persist, where the value of waiting is explicitly an "option to learn". If the firm invests now it gets the benefit of the cash flow $\bar{X}(p)$ over the next instant. This is the term on the left hand side of the equal sign. If the firm waits a moment and invests only if all of the past shocks persist, it faces a small chance of one of the shocks reversing, in which case the expected value gained by waiting is equal to first term on the right hand side. Specifically, $G\left(\frac{\bar{X}(p)}{1+\varphi}, \tilde{p}(p)\right) + I - S\left(\frac{\bar{X}(p)}{1+\varphi}, \tilde{p}(p)\right)$ is the value gained by not investing should an existing shock reverse over the next instant, while $(\lambda_3 \sum_{i=1}^{\infty} p_i i)$ is the likelihood of this occurring. The second term on the right hand side of the equal sign, $rI$, is the savings from delaying the investment cost by an instant. The third term is the convexity term which represents the traditional option to wait given Brownian uncertainty. At the optimal trigger, $\bar{X}(p)$, these two sides are exactly equal, and the firm is indifferent between investing now or a moment later.

We can now summarize the solution to the optimal investment timing problem in this multi-shock setting.

**Proposition 2.** *The optimal investment strategy for the model outlined in this section is to invest when $X(t)$ exceeds $\bar{X}(p(t))$ for the first time.*

To provide intuition for the multi-shock extension, Fig. 5 plots a simulated path of the cash flow process $X(t)$ along with the corresponding investment trigger $\bar{X}(p(t))$. In order to accentuate the impact of multiple shocks, for this numerical solution we consider a pure-jump process by setting $\alpha = \sigma = 0$. In this example, each shock increases the value of the cash flows by 5%. A new permanent shock occurs, on average, every two years ($\lambda_1 = 0.5$). A new temporary shock occurs, on average, every year ($\lambda_2 = 1$) and takes six months to revert ($\lambda_3 = 2$). The dynamics of the cash flow process is very straightforward. In the 5-year period there are four arrivals of new shocks, corresponding to upward jumps in $X(t)$. Two out of four shocks revert corresponding to downward jumps in $X(t)$. The dynamics of the investment boundary is more interesting. Prior to the arrival of the first shock, there is no uncertainty about past shocks. Since there is no Bayesian updating at this time, the trigger is constant at $\bar{X}(\mathbf{0})$. After the first shock arrives, the trigger jumps up to $\bar{X}(p_0, 0, ...)$ as the firm becomes unsure if the outstanding shock is permanent or temporary. As time goes by and the cash flow process does not revert back, the first shock is more likely to be permanent. As a result, the firm's beliefs become more optimistic, and the investment trigger decreases. This intuition underlies the whole dynamics of the trigger in Fig. 5. An arrival of a new shock leads to an upward jump in the trigger due to an increase in uncertainty. When the cash flow process is stable, the trigger goes down as the firm updates its beliefs about the past shocks. When an outstanding shock reverses, the firm learns for sure that one of the outstanding shocks is temporary. Hence, reversal leads to a downward jump in the trigger. The investment occurs when the $X(t)$ exceeds the investment trigger $\bar{X}(p(t))$ for the first time. In the example in Fig. 5, this happens at $t = 2.7$ when the firm becomes sufficiently sure that the outstanding shock is permanent.

By allowing for multiple shocks, the model of this section illuminates the nature of the Bayesian investment problem beyond that of the simpler one-shock model. Most notably, the exercise trigger, $\bar{X}(p(t))$, is a multi-dimensional function, demonstrating its dependence on all persisting past shocks. Thus, as illustrated in Fig. 5, the investment trigger jumps by a discrete amount at all moments when either a new jump occurs or an existing shock reverses. In between such jumps, the trigger declines as Bayesian uncertainty is reduced.

# 6 Conclusion

This paper studies the optimal timing of investment in the presence of uncertainty about both future and past shocks. We augment the standard Brownian uncertainty driving traditional real options models with additional Bayesian uncertainty over distinguishing between the temporary or permanent nature of past cash flow shocks. As a result, the evolving uncertainty is no longer constant, and is driven by Bayesian updating, or learning. We solve for the optimal investment rule and show that the implied investment behavior differs significantly from that predicted by standard real options models. For example, in contrast to the standard real options implications, firms hold both a traditional Brownian "option to wait" as well as a Bayesian "option to learn", investment may occur at a time of stable or decreasing cash flows, investment may respond sluggishly to positive cash flow shocks, and investment timing will critically depend on the maturity structure of the project cash flows.

Several further extensions of the model would prove interesting. First, the model could be extended to a multiple firm industry equilibrium in the manner of Grenadier (2002). Of particular interest would be the fact that firms need to update their beliefs about not only their own past shocks, but also those of their competitors. Second, the structure of the jump component could be made richer by permitting more than just either permanent or temporary shocks. For example, shocks could have a greater variety of (unobserved) types, such as $n$ types with reversion intensities $\lambda_1^{reversion}, \lambda_2^{reversion}, ..., \lambda_n^{reversion}$.

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# Appendix

## Proofs and Derivations

**Proof of Section 3.2 conjecture that $\bar{X}(p) < (1+\varphi)X^*$.**

Let us modify the problem in the following way. Suppose that while the firm waits, the jump cannot revert back, and after the firm invests, it reverts back in the same way as in the original problem. Since for any $(X, p)$ this modification does not affect the value of immediate investment, but increases the value of waiting relative to the initial problem, the corresponding investment trigger $\bar{X}_{\text{mod}}(p)$ is higher than $\bar{X}(p)$ for any $p \in (0, p_0]$. Since the jump cannot revert back before investment, the firm does not update its beliefs in the modified problem. As a result, the investment trigger in the modified problem can be explicitly computed:

$$\bar{X}(p) < \bar{X}_{\text{mod}}(p) = \frac{\beta}{\beta-1}\frac{r-\alpha}{1-p\frac{\lambda_3\varphi}{(r-\alpha+\lambda_3)(1+\varphi)}}I < (1+\varphi)X^*. \tag{59}$$

This implies that indeed, it is optimal for the firm to invest at a trigger that is below $(1+\varphi)X^*$.

**Closed form solutions for the model of Section 3.2 when $\alpha > 0$ and $\sigma = 0$.**

While the exercise trigger $\bar{p}(X)$ is characterized by (29), it is not solvable in closed-form, since the value function $G(X, p)$ itself is not available in closed-form. However, for the special case in which $\sigma = 0$, $\alpha \geq 0$, the closed-form solution for the trigger is

$$\bar{p}(X)\,|_{\sigma=0} = \frac{X - rI}{\lambda_3\left[I + \left(\frac{X}{rI(1+\varphi)}\right)^{\frac{r}{\alpha}}\frac{\alpha I}{r-\alpha} - \frac{X}{(1+\varphi)(r-\alpha)}\right]}. \tag{60}$$

The corresponding value of the investment option equals

$$G(X,p)\,|_{\sigma=0} = p\frac{\alpha I}{r-\alpha}\left(\frac{X}{(1+\varphi)rI}\right)^{\frac{r}{\alpha}} + (1-p)X^{\frac{r}{\alpha}}\Gamma\left(X\left(\frac{1}{p}-1\right)^{-\frac{\alpha}{\lambda_3}}\right), \tag{61}$$

where

$$\begin{aligned}\Gamma(y) = {} & \left(\tfrac{\lambda_1}{\lambda_2}\right)^{\frac{\alpha}{\lambda_3}} y \left(\tfrac{\lambda_1}{r-\alpha} e^{(\alpha-r)t^*\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\alpha}{\lambda_3}} y\right)} + \lambda_2 \tfrac{1+\varphi+\frac{\lambda_3}{r-\alpha}}{(r+\lambda_3-\alpha)(1+\varphi)} e^{(\alpha-r-\lambda_3)t^*\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\alpha}{\lambda_3}} y\right)}\right) \\ & - \left(\tfrac{y}{(1+\varphi)rI}\right)^{\frac{r}{\alpha}} \tfrac{\alpha I}{r-\alpha} \left(\tfrac{\lambda_1}{\lambda_2}\right)^{\frac{r}{\lambda_3}} \lambda_2 e^{-\lambda_3 t^*\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\alpha}{\lambda_3}} y\right)},\end{aligned} \tag{62}$$

where $t^*(z)$ is a function defined implicitly by

$$\frac{\lambda_2\lambda_3}{\lambda_1 e^{\lambda_3 t^*} + \lambda_2} = \frac{ze^{\alpha t^*} - rI}{\left(\frac{z}{(1+\varphi)rI}\right)^{\frac{r}{\alpha}}\frac{\alpha I e^{rt^*}}{r-\alpha} + I - \frac{ze^{\alpha t^*}}{(r-\alpha)(1+\varphi)}}. \tag{63}$$

Notice that when $\alpha = 0$, investment does not occur when the jump reverts. Therefore, in this case, the trigger (60) coincides with (15)[23].

**Proof of Section 3.3 conjecture that $\hat{X} \geq \bar{X}(p_0)/(1+\varphi)$.**

By contradiction, suppose that it is optimal to invest at some trigger $X'$ below $\bar{X}(p_0)/(1+\varphi)$. Let us modify the problem in the following way. Suppose that the upward jump occurs immediately, that is, $\lambda_1 + \lambda_2 = +\infty$ with $p_0$ being unchanged. Notice that for each sample path the project in the modified problem yields the same cash flows as the project in the original problem with the difference that the range of extra cash flows generated by the upward shock occurs earlier. Hence, investment in the modified problem occurs earlier than in the original problem. In particular, since it was optimal to invest at $X'$ in the original problem, it is optimal at $X'$ in the modified problem. Since in the modified problem the jump occurs immediately, the value of the investment option is $G(X(1+\varphi), p_0)$. However, from the previous subsection we know that it is strictly optimal to wait for all $X(1+\varphi) < \bar{X}(p_0)$. Therefore, it cannot be optimal to invest at any $X'$. This implies that indeed, it is strictly optimal to wait for any $X$ below $\bar{X}(p_0)/(1+\varphi)$. Therefore, the optimal investment policy is characterized by the critical value (40) at which it is optimal to invest.

**Derivation of the investment trigger $\hat{X}$ and the investment option value $F(X)$ for the model with Brownian uncertainty.**

The general solutions to Eq. (33) and (34) are given by

$$F_L(X) = C_1 X^{\gamma_1} + C_2 X^{\gamma_2} + \frac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)}\left(\Gamma_2(X) - \Gamma_1(X)\right), \tag{64}$$

$$F_H(X) = A_1 X^{\gamma_1} + A_2 X^{\gamma_2} + \frac{(\lambda_1+\lambda_2)\left((1+\varphi)(r-\alpha)+\lambda_3\right)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)}X - \frac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2}I, \tag{65}$$

where $\Gamma(X)$ is given by (41) and

$$\Lambda(X) = X^{\gamma_1}\int \frac{G(X(1+\varphi), p_0)}{X^{\gamma_1+1}}dX. \tag{66}$$

We have five boundary conditions (35)-(39) to determine four unknown constants ($A_1$, $A_2$, $C_1$ and $C_2$) and the investment trigger $\hat{X}$. The fifth boundary condition implies that

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[23] Note that since in (15) $X_0$ denotes the level of $X(t)$ before the positive jump occurred, we need to use $\bar{p}(X_0(1+\varphi))$ to ensure equivalence.

$C_2 = 0$. The other four boundary conditions give the following system of equations:

$$\begin{gathered}
A_1\hat{X}^{\gamma_1} + A_2\hat{X}^{\gamma_2} = \tfrac{\hat{X}}{r-\alpha+\lambda_1+\lambda_2} - \tfrac{rI}{r+\lambda_1+\lambda_2} \\
\gamma_1 A_1\hat{X}^{\gamma_1} + \gamma_2 A_2\hat{X}^{\gamma_2} = \tfrac{\hat{X}}{r-\alpha+\lambda_1+\lambda_2} \\
A_1\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)^{\gamma_1} + A_2\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)^{\gamma_2} + \tfrac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)}\tfrac{\bar{X}(p_0)}{1+\varphi} - \tfrac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2}I \\
= C_1\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)^{\gamma_1} + \tfrac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)}\left(\Gamma\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right) - \Lambda\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)\right) \\
\gamma_1 A_1\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)^{\gamma_1} + \gamma_2 A_2\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)^{\gamma_2} + \tfrac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)}\tfrac{\bar{X}(p_0)}{1+\varphi} \\
= \gamma_1 C_1\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)^{\gamma_1} + \tfrac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)}\left(\gamma_2\Gamma\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right) - \gamma_1\Lambda\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)\right),
\end{gathered} \tag{67}$$

Combining the last two equations, we get

$$\begin{aligned}
A_2 = \left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)^{-\gamma_2} \Big[ & \tfrac{1-\gamma_1}{\gamma_1-\gamma_2}\tfrac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)}\tfrac{\bar{X}(p_0)}{1+\varphi} \\
& + \tfrac{\gamma_1}{\gamma_1-\gamma_2}\tfrac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2}I + \tfrac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)}\Gamma\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)\Big].
\end{aligned} \tag{68}$$

Combining the first two equations, we get

$$\hat{X} = \frac{\gamma_1}{\gamma_1-1}\frac{(r-\alpha+\lambda_1+\lambda_2)\,rI}{r+\lambda_1+\lambda_2} + \frac{\gamma_1-\gamma_2}{\gamma_1-1}A_2\hat{X}^{\gamma_2}. \tag{69}$$

Plugging (68) into (69) yields the expression for the investment trigger (40).

The corresponding value of the investment opportunity is

$$F(X) = \begin{cases}
C_1X^{\gamma_1} + \tfrac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)}\left(\Gamma(X) - \Lambda(X)\right), & X \leq \tfrac{\bar{X}(p_0)}{1+\varphi} \\
A_1X^{\gamma_1} + A_2X^{\gamma_2} + \tfrac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)}X - \tfrac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2}I, & \tfrac{\bar{X}(p_0)}{1+\varphi} \leq X \leq \hat{X} \\
\tfrac{X}{r-\alpha} + \tfrac{\varphi X}{r-\alpha+\lambda_1+\lambda_2}\left(\tfrac{\lambda_1}{r-\alpha} + \tfrac{\lambda_2}{r-\alpha+\lambda_3}\right) - I, & X \geq \hat{X},
\end{cases} \tag{70}$$

where $A_2$ is given by (68), and $A_1$ and $C_1$ satisfy

$$A_1 = \hat{X}^{-\gamma_1}\left(\frac{\hat{X}}{r-\alpha+\lambda_1+\lambda_2} - \frac{rI}{r+\lambda_1+\lambda_2} - A_2\hat{X}^{\gamma_2}\right), \tag{71}$$

$$\begin{aligned}
C_1 = A_1 + A_2\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)^{\gamma_2-\gamma_1} + \left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)^{-\gamma_1}\Big( & \tfrac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)}\tfrac{\bar{X}(p_0)}{1+\varphi} \\
& - \tfrac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2}I + \tfrac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)}\left(\Lambda\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right) - \Gamma\left(\tfrac{\bar{X}(p_0)}{1+\varphi}\right)\right)\Big).
\end{aligned} \tag{72}$$

# Numerical Procedures

**Numerical procedure for computing $\bar{X}(p)$ in Section 3.**

To compute the trigger functions we use a variation of the least-squares method developed by Longstaff and Schwartz (2003). Note that when $p = 0$, the model becomes standard, so $\bar{X}(0) = X^*$. Also, note that $p(T) \to 0$ as $T \to \infty$, where $T$ is the time that passes after the arrival of the shock. Because of that, we can approximate $\bar{X}(p(T))$ for some large $T$ by $\bar{X}(0)$. After that, take some small $\Delta$, and compute $p(T-\Delta)$ from (8). Then, use the least squares method of Longstaff and Schwartz (2003) to estimate the second derivative of the conditional expected payoff from waiting at time $T-\Delta$ until time $T$. Then, use this estimate and (29) to compute $\bar{X}(p(T-\Delta))$. We repeat this $N$ times for a sufficiently large $N$ such that $p(T-N\Delta) > p_0$. More specifically, at any step $n$:

1. Use $\bar{X}(p(T-k\Delta))$, $k = 0, 1, ..., n-1$ and the least squares method to estimate the second derivative of the conditional expected payoff from waiting at time $T-n\Delta$;
2. Use this estimate as input in (29) to compute $\bar{X}(p(T-n\Delta))$.

**Numerical procedure for computing $\bar{X}(p)$, $p = (p_1, p_2, 0, ..., 0, ...)'$ in Section 5.**

First, we compute $S(X,p)$. Conjecture that $S(X,p)$ is given by

$$S(X,p) = a_0 X + \sum_{i=1}^{\infty} p_i (a_i - a_0) X \tag{73}$$

for some constants $a_0, a_1, ....$ Plugging (73) into (52), we get

$$\begin{array}{c} \left(r+\lambda_1+\lambda_2+\lambda_3 \sum_i i p_i\right) a_0 + (r+\lambda_1+\lambda_2) \sum_i (a_i - a_0) p_i = \\ = (\lambda_1+\lambda_2)(1+\varphi)\left(p_0 a_1 + (1-p_0) a_0 + \sum_i \left[p_0 (a_{i+1} - a_1) + (1-p_0)(a_i - a_0)\right] p_i\right) \\ + \frac{\lambda_3}{1+\varphi} \sum_i a_{i-1} i p_i - \lambda_3 \sum_i (a_i - a_0) i p_i + 1. \end{array} \tag{74}$$

This equation must hold for any $p$. This happens if and only if coefficients with 1, $p_1$,$p_2$,... on the left hand and right hand side are equal. Matching the coefficients, we get

$$a_k = \begin{cases} \frac{(1+\varphi)\lambda_2}{r+\lambda_2-\varphi\lambda_1} a_1 + \frac{1}{r+\lambda_2-\varphi\lambda_1} \text{ for } k=0, \\ \frac{1}{r+\lambda_2-\varphi\lambda_1+\lambda_3 k} + \frac{\lambda_3 k}{(1+\varphi)(r+\lambda_2-\varphi\lambda_1+\lambda_3 k)} a_{k-1} + \frac{(1+\varphi)\lambda_2}{r+\lambda_2-\varphi\lambda_1+\lambda_3 k} a_{k+1} \text{ for } k = 1, 2, ... \end{cases} \tag{75}$$

Hence, coefficients $a_0$,$a_1$,... are defined as solutions to this recurrence relation subject to the boundary condition $\lim_{k\to\infty} a_k = 0$. This system of equations is solved numerically.

Second, we compute $\bar{X}(p)$ for $p = (p_1, 0, ..., 0, ...)$. In this case, $\bar{X}(p)$ if found as a solution to

$$\bar{X}(p) = \lambda_3 p_1 \left[G\left(\frac{\bar{X}(p)}{1+\varphi}, 0\right) + I - S\left(\frac{\bar{X}(p)}{1+\varphi}, 0\right)\right] + rI,$$

$$G\left(\frac{\bar{X}(p)}{1+\varphi}, 0\right) = \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+r} G\left(\bar{X}(p), (p_0, 0, ..., 0, ...)'\right).$$

If $p_1$ is above $p_0$, then

$$G\left(\bar{X}\left(p\right),\left(p_0,0,...,0,...\right)'\right)=S\left(\bar{X}\left(p\right),\left(p_0,0,...,0,...\right)'\right)-I.$$

If $p_1$ is below $p_0$, then $G\left(\bar{X}\left(p\right),\left(p_0,0,...,0,...\right)'\right)$ is computed by simulations. In this case, investment occurs at the first instant when an upward jump arrives or $p\left(t\right)$ reduces from $p_0$ to $p$.

Finally, we use these results to approximate $\bar{X}\left(p\right)$ for $p=\left(p_1,p_2,0,...,0,...\right)$ using the following procedure. First, we take $\bar{X}\left(p\right)$ to be equal to $\bar{X}\left(\tilde{p}\right)$, where $\tilde{p}=\left(\sum_{i=1}^{\infty}p_i,0,...,0,...\right)$. Using this threshold, we simulate the option values $G\left(X,p\right)$ for $p=\left(p_1,...,p_{M-1},0,...,0,...\right)$. Then, we use the simulated option values as input in (58) to re-evaluate $\bar{X}\left(p\right)$ for $p=\left(p_1,...,p_M,0,...,0,...\right)$. After that, we take $\bar{X}\left(p\right)$ to be equal to $\bar{X}\left(\tilde{p}\right)$, where $\tilde{p}=\left(p_1,...,p_{M-1},\sum_{i=M}^{\infty}p_i,0,...,0,...\right)$. Using this threshold, we again simulate the option values $G\left(X,p\right)$ for $p=\left(p_1,...,p_{M-1},0,...,0,...\right)$. Then, we use the simulated option values as input in (58) to re-evaluate $\bar{X}\left(p\right)$ for $p=\left(p_1,...,p_M,0,...,0,...\right)$. This is repeated until convergence. If $M\rightarrow\infty$, this procedure should converge to the true threshold $\bar{X}\left(p\right)$. Since the procedure is computationally intensive, we are able to use only the maximum of $M=5$. We use the resulting $\bar{X}\left(p\right)$ for $p=\left(p_1,p_2,0,...,0,...\right)$ to plot the simulated sample paths in Section 5.

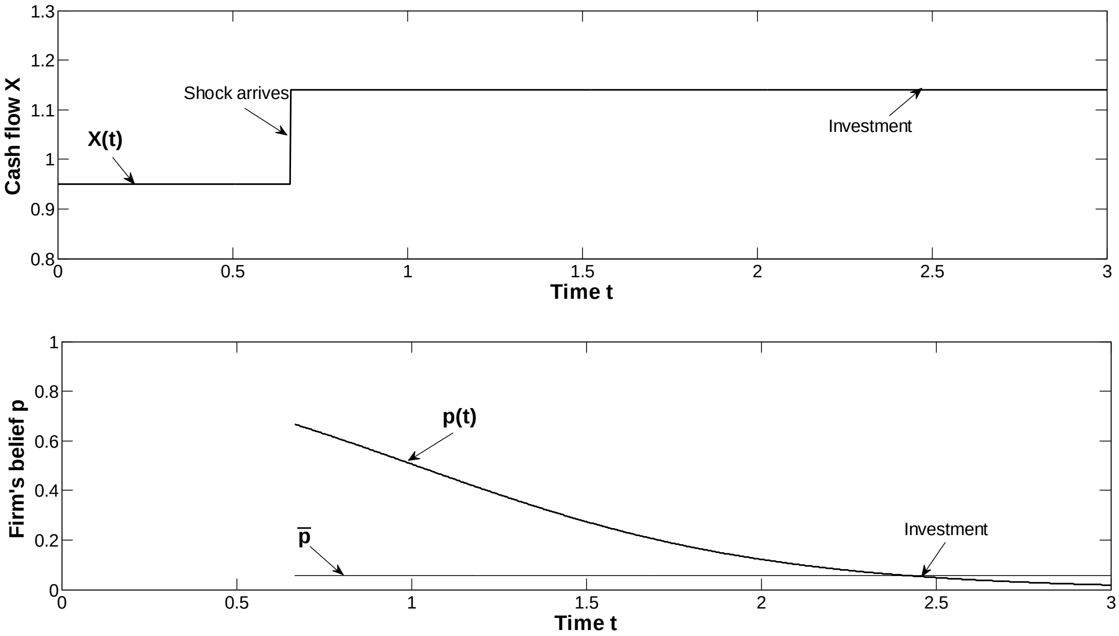

**Figure 1. Simulation of the firm's investment strategy.** The top graph shows the simulated sample path of the cash flow process, X(t), for the simple model of Section 2.3. An upward step corresponds to the arrival of a shock. The bottom graph shows the dynamics of the firm's belief process as well as the investment threshold $\bar{p}$. The investment occurs when the firm's belief process falls to $\bar{p}$. The parameter values are r=0.04, $\varphi$=0.2, $\lambda_1$=0.5, $\lambda_2$=1, $\lambda_3$=2, I=25, and X=0.95.

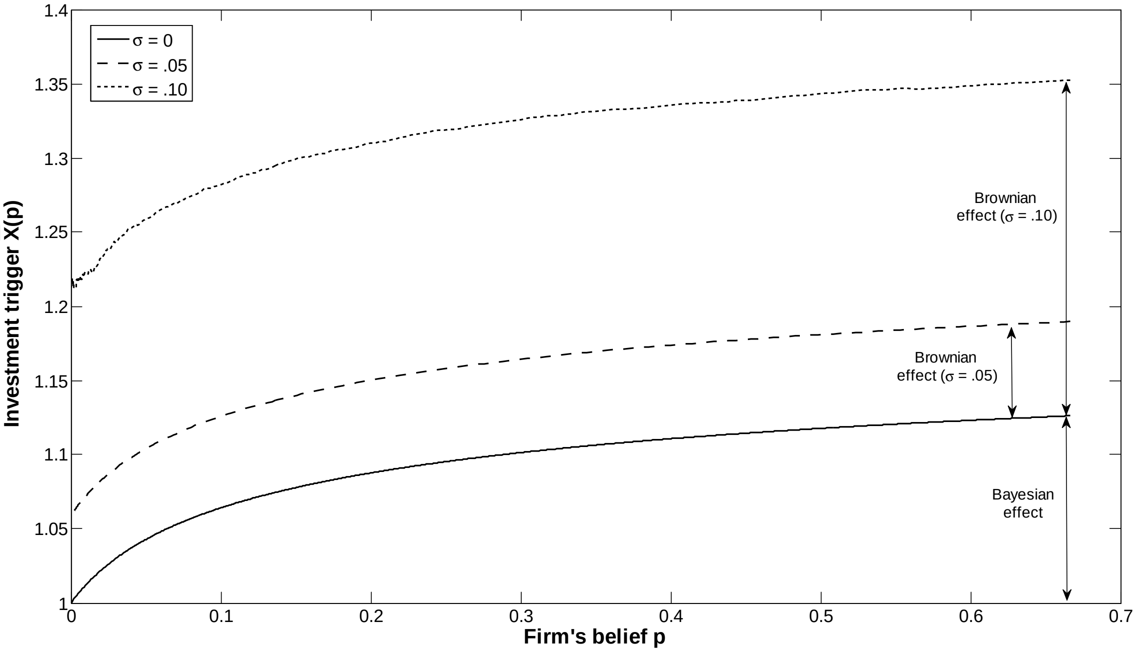

**Figure 2. Investment trigger functions for different values of the Brownian volatility parameter.** The graph plots the investment trigger function $\bar{X}(p)$ for different values of the Brownian volatility parameter $\sigma$. The bottom curve corresponds to the case of pure Bayesian uncertainty ($\sigma$=0). The middle and the top curves correspond to the cases of both Bayesian and Brownian uncertainties ($\sigma$=0.05 and $\sigma$=0.10, respectively). As a result, the change of the trigger along each curve is due to the impact of Bayesian uncertainty, while the upward shift of the whole trigger function is due to the impact of Brownian uncertainty. The parameter values are $r$=0.04, $\alpha$=0.02, $\varphi$=0.2, $\lambda_1$=0.5, $\lambda_2$=1, $\lambda_3$=2, and $I$=25.

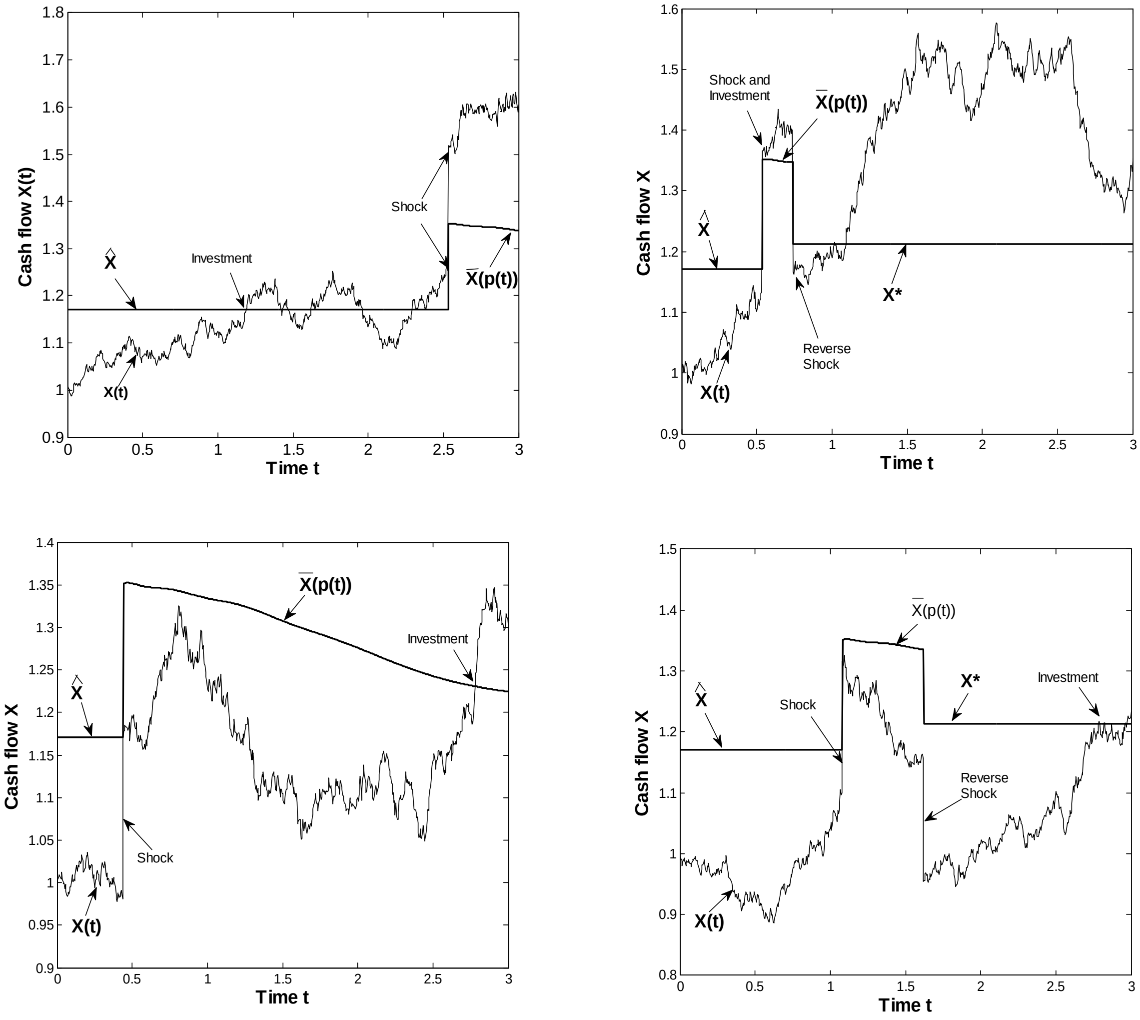

**Figure 3. Simulations of the firm's investment strategies.** The figure shows the simulated paths of the cash flow process, X(t), (plotted in thin lines) and the corresponding investment triggers (plotted in bold lines) for four different scenarios specified in Proposition 1. The optimal exercise strategy is to invest at the first time when X(t) reaches the investment trigger for the first time. The parameter values are r=0.04, $\alpha$=0.02, $\sigma$=0.10, $\varphi$=0.2, $\lambda_1$=0.5, $\lambda_2$=1, $\lambda_3$=2, I=25, and X(0)=1.

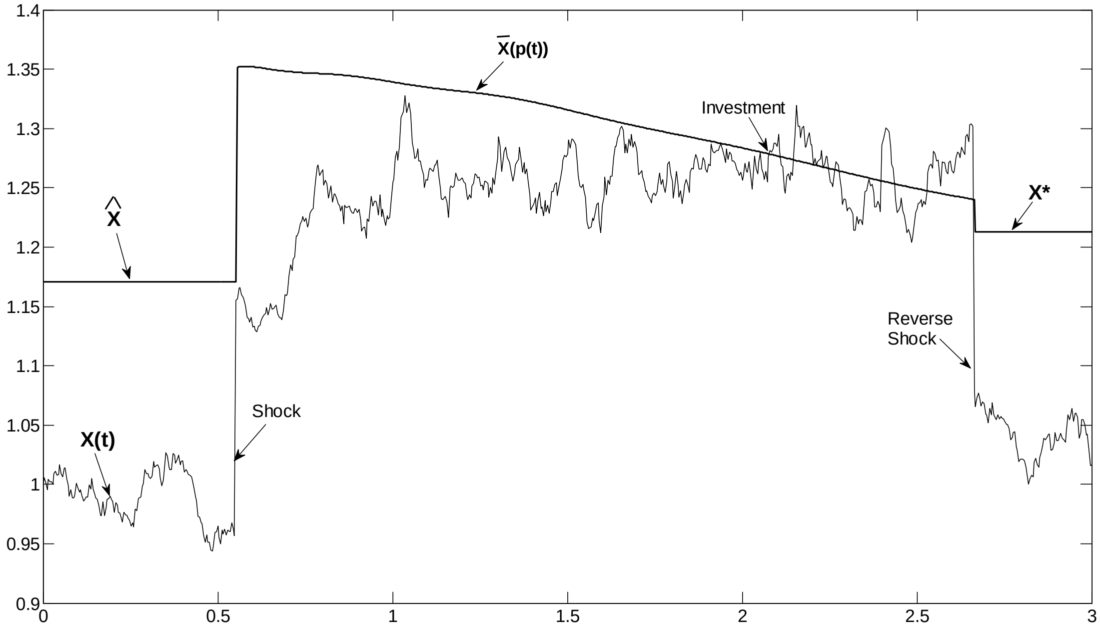

**Figure 4. Investment in the Face of Stable Cash Flows: Illustration.** The figure shows the simulated path of the cash flow process, X(t), (plotted in the thin line) and the corresponding investment trigger (plotted in the bold line). The optimal exercise strategy is to invest at the first time when X(t) reaches the investment trigger for the first time. The parameter values are r=0.04, $\alpha$=0.02, $\sigma$=0.10, $\varphi$=0.2, $\lambda_1$=0.5, $\lambda_2$=1, $\lambda_3$=2, I=25, and X(0)=1.

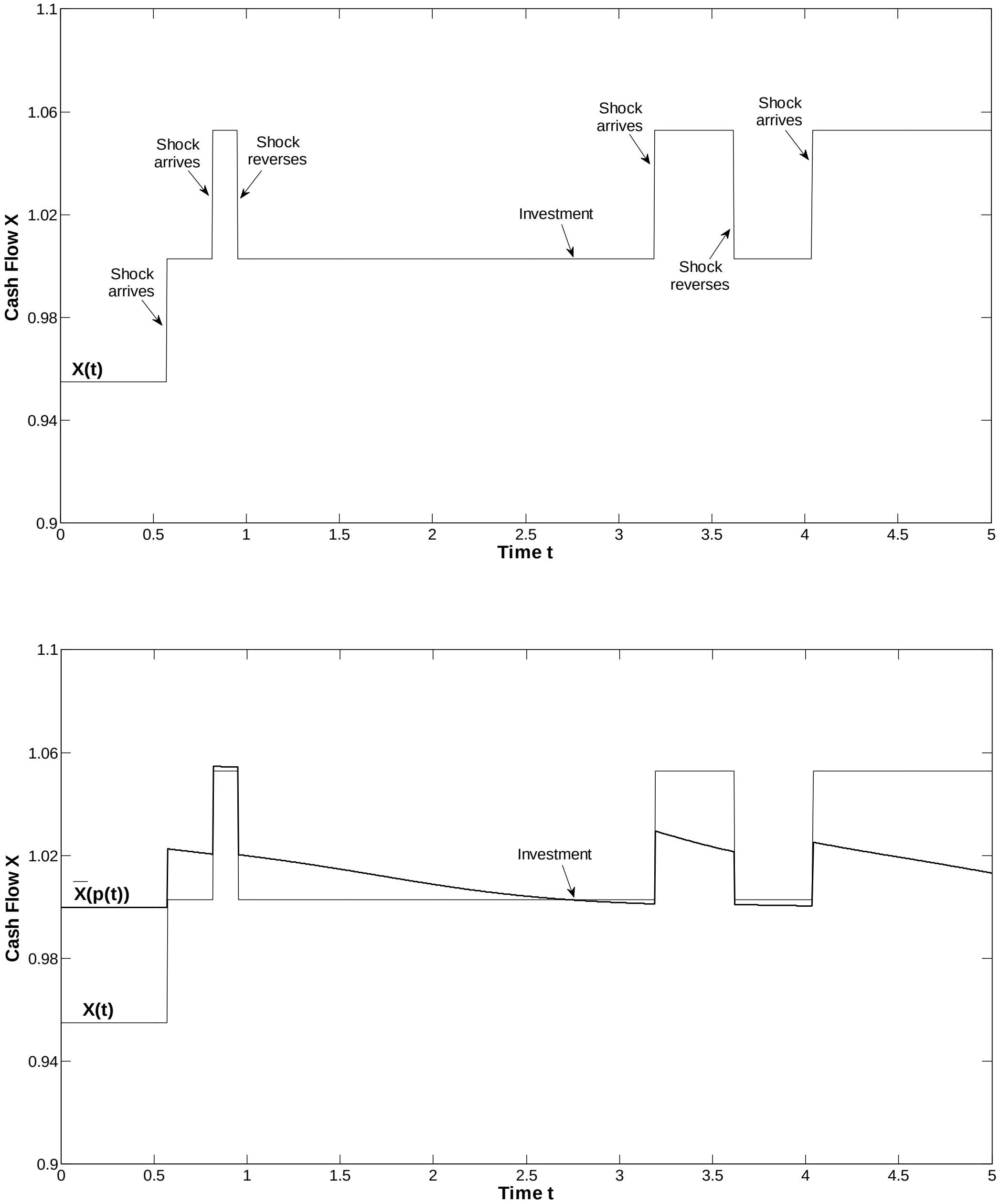

**Figure 5. Simulation of the firm's investment strategy.** The top graph shows the simulated sample path of the cash flow process, X(t), for the model of Section 5. Each of the four upward steps corresponds to the arrival of a new shock. Similarly, each of the two downward steps corresponds to the reversal of an outstanding shock. The bottom graph adds the investment trigger $\bar{X}$(p(t)) to the simulated sample path from the top graph. The optimal investment strategy is to invest at the first time when X(t) reaches the investment trigger. The parameter values are r=0.04, $\alpha$=0, $\sigma$=0, $\varphi$=0.05, $\lambda_1$=0.5, $\lambda_2$=1, $\lambda_3$=2, I=25, and X(0)=0.955.